Characteristic oscillations of phase properties for pair coherent states in the two-mode Jaynes-Cummings-model dynamics

S.-C. Gou
Department of Physics, National Tsing Hua University, Hsinchu, Taiwan 30043, Republic of China
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The phase properties of pair coherent states in the two-mode Jaynes-Cummings model are investigated based on the Pegg-Barnett phase formalism. The general expressions of the properties for the phase-sum operator of the two correlated fields are presented, in which the Stark shifts are considered. Analytic results are given in the strong-field approximations. It is found that the phase properties such as the variance of the phase-sum operator and the variances of the cosine and the sine of the phase-sum operator exhibit characteristic periodic oscillations in which the period is determined solely by the Stark shifts.

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I. INTRODUCTION

The Jaynes-Cummings model (JCM) [1], which describes the interaction of two-level atom with single mode of the quantized field, is considered to be one of the most successful models in quantum optics. Several nonclassical phenomena, such as the spontaneous collapses and revivals of the atomic population, sub-Poissonian photon distribution, and field squeezing have been predicted and discussed theoretically [2–6]. Recent advances in experimental techniques have also made it possible to realize the system inside a superconducting cavity, which led to the observations of the aforementioned features [7].

In addition to the standard JCM, some generalized models [8–11] have been constructed and extensively studied. One of these generalizations (multilevels, multiphotons) is to replace the mediated photon by a nondegenerate photon pair, i.e., photons of different modes are either emitted or absorbed in pairs by the atom. This generalized JCM has been studied by several authors [10–12]. Similar novel quantum features, such as the collapses and revivals of the Rabi oscillations and sub-Poissonian photon distributions, have also been found. It is shown that the quantum dynamics of the two-mode JCM depend crucially on the initial two-mode field states. Especially for some nonclassical two-mode states such as the pair coherent states and other SU(1,1) correlated states, the quantum collapse and revival characteristics of the two-mode JCM exhibit qualitatively different behavior from those in the uncorrelated two-mode field states [10–12].

Recently, Pegg and Barnett [13,14] have introduced a new set of formalisms to define a Hermitian phase operator, and this has renewed considerable interest in studying the phase properties of fully quantized radiation fields. Using the Pegg-Barnett approach, phase properties of special field states such as the squeezed states [15], pair coherent states [16], and other nonclassical ones [17,18] have been reported. The same approach can also be employed to the JCM and other related systems [19–21]. In this paper, we use the Pegg-Barnett approach to study the phase dynamics of pair coherent states in the nondegenerate two-photon JCM. The joint probability distribution for the phases of the two modes is obtained and shown to depend explicitly on the sum of the two phases. The variance of the phase-sum operator and the variances of the cosine and sine of the phase-sum operator are investigated. In particular, the influences of Stark shifts on these phase properties are also studied. The organization of this paper is as follows: In Sec. II, the properties of the pair coherent states are described briefly. In Sec. III, the field dynamics of the two-mode JCM are presented, in which the Stark shifts are considered. In Sec. IV, the time evolution of the aforementioned phase properties for pair coherent states are investigated. In the strong-field approximations, some analytic results are shown and discussed. Finally, the concluding remarks are given in Sec. V.

II. FEATURES OF THE PAIR COHERENT STATES

The pair coherent states are closely related to the SU(1,1) correlated states, which may be regarded as a special type of SU(1,1) coherent states according to the definition of Barut and Girardello [22] [the eigenstates of the SU(1,1) lower operator]. Let $\hat{a}_1$ ($\hat{a}_1^\dagger$) and $\hat{a}_2$ ($\hat{a}_2^\dagger$) be the annihilation (creation) operators of modes 1 and 2, respectively. Therefore, $\hat{a}_1\hat{a}_2$ ($\hat{a}_1^\dagger\hat{a}_2^\dagger$) stands for the pair annihilation (creation) operator for the two modes. The pair coherent states, denoted by $|z,q\rangle$, are defined as eigenstates of the pair annihilation operator and the number difference operator, i.e.,

$$\hat{a}_1\hat{a}_2|z,q\rangle = z|z,q\rangle \tag{2.1}$$

and

$$(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_2)|z,q\rangle = q|z,q\rangle\ , \tag{2.2}$$

where z is a complex number and q is the degeneracy parameter. Specifically, the pair coherent states can be expanded as superpositions of the two-mode Fock states, i.e.,

$$|z,q\rangle = \mathcal{N}_q \sum_{n=0}^{\infty} \frac{z^n}{[n!(n+q)!]^{1/2}} |n+q,n\rangle\ . \tag{2.3}$$

The normalized constant $\mathcal{N}_q$ is given by

$$\mathcal{N}_q^2=\left[\sum_{n=0}^{\infty}\frac{|z|^{2n}}{n!(n+q)!}\right]^{-1}=\frac{|z|^q}{I_q(2|z|)}, \tag{2.4}$$

where I_q is the modified Bessel function of the first kind of order q.

The probability of finding n photons in mode 2 and $n+q$ photons in mode 1 is

$$P_n=|\langle n,n+q|z,q\rangle|^2=\mathcal{N}_q^2\frac{|z|^{2n}}{n!(n+q)!}, \tag{2.5}$$

which is sub-Poissonian. Accordingly, the mean photon numbers $\langle\hat{n}_1\rangle$ and $\langle\hat{n}_2\rangle$ for $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\langle\hat{n}_1\rangle=q+\langle\hat{n}_2\rangle, \tag{2.6}$$

$$\langle\hat{n}_2\rangle=\mathcal{N}_q^2\sum_{n=0}^{\infty}\frac{n|z|^{2n}}{n!(n+q)!}=\frac{I_{q+1}(2|z|)}{I_q(2|z|)}|z|. \tag{2.7}$$

In addition to the sub-Poissonian statistics, the pair coherent states also possess other nonclassical features, such as the correlation in the number fluctuations, squeezing, and violations of Cauchy-Schwartz inequalities. For detailed descriptions of these properties, see Ref. [23].

III. FIELD DYNAMICS OF THE TWO-MODE JCM

The system considered is an effective two-level atom, in which the transition between the excited state $|e\rangle$ and the ground state $|g\rangle$ is mediated by the two-cavity modes with frequencies Ω_1 and Ω_2, respectively.

In the two-photon processes, an intermediate state $|i\rangle$ is involved which is assumed to be coupled to $|e\rangle$ and $|g\rangle$ by dipole-allowed transitions with strengths g_1 and g_2. Let $\omega_g<\omega_i<\omega_e$, where ω_g,ω_e, and ω_i denote the corresponding frequencies of the atomic energy levels $|g\rangle$, $|e\rangle$, and $|i\rangle$, respectively. Consider the exact two-photon resonance, i.e., $\omega_i-\omega_g=\Omega_1-\Delta$, $\omega_e-\omega_i=\Omega_2+\Delta$. If the detuning $|\Delta|$ is assumed to be off resonance from one photon linewidth, then the intermediate state $|i\rangle$ can be eliminated adiabatically and the effective Hamiltonian of the two-level atom can be written in the rotating-wave approximation as [24]

$$\hat{H}_{\text{eff}}=\sum_{i=1,2}\Omega_i\hat{a}_i^\dagger\hat{a}_i+\frac{\omega}{2}(\hat{S}_{ee}-\hat{S}_{gg})+\beta_1\hat{S}_{gg}\hat{a}_1^\dagger\hat{a}_1+\beta_2\hat{S}_{ee}\hat{a}_2^\dagger\hat{a}_2+g(\hat{a}_1\hat{a}_2\hat{S}_{eg}+\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{S}_{ge}), \tag{3.1}$$

where $\omega=\omega_e-\omega_g,\beta_1=g_1^2/|\Delta|,\beta_2=g_2^2/|\Delta|$, and $g=g_1g_2/|\Delta|$. Here $\hat{a}_i^\dagger$ $(\hat{a}_i)$ is the creation (annihilation) operator of the cavity modes, and $\hat{S}_{jk}=|j\rangle\langle k|$ is the atomic operator. This Hamiltonian describes the two-photon transition between $|g\rangle$ and $|e\rangle$. The Stark shifts are characterized by the effective coupling constants β_1 and β_2 that give rise to the intensity-dependent energy shifts of the two atomic levels.

For the subspace expanded by $|e,n+q,n\rangle$ and $|g,n+q+1,n+1\rangle$, the eigenvalues of the corresponding irreducible representation are

$$\lambda_{1,2}(n,q)=A_n^q-\tfrac{1}{2}D_n^q\pm Q_n^q, \tag{3.2}$$

where

$$\begin{aligned}A_n^q&=\Omega_1(n+q)+\Omega_2 n+\tfrac{1}{2}\omega+\beta_2 n,\\ D_n^q&=\beta_2 n-\beta_1(n+q+1)=g\left[\frac{n}{r}-r(n+q+1)\right],\\ Q_n^q&=[\tfrac{1}{4}(D_n^q)^2+g^2(n+q+1)(n+1)]^{1/2},\end{aligned} \tag{3.3}$$

and $r=g_1/g_2$.

Neglecting the free evolution terms in Eq. (3.2), which give trivial phase factors only, the matrix representation of the time evolution operator is expressed as

$$\hat{U}(n,q;t)=\begin{bmatrix}U_{ee}(n,q;t) & U_{eg}(n,q;t)\\ U_{ge}(n,q;t) & U_{gg}(n,q;t)\end{bmatrix}, \tag{3.4}$$

where the matrix elements are

$$\begin{aligned}U_{ee}(n,q;t)&=\cos(Q_n^q t)-\frac{iD_n^q}{2Q_n^q}\sin(Q_n^q t),\\ U_{gg}(n,q;t)&=\cos(Q_n^q t)+\frac{iD_n^q}{2Q_n^q}\sin(Q_n^q t),\\ U_{eg}(n,q;t)&=-i\frac{[(Q_n^q)^2-(D_n^q)^2/4]^{1/2}}{Q_n^q}\sin(Q_n^q t),\\ U_{ge}(n,q;t)&=U_{eg}(n,q;t).\end{aligned} \tag{3.5}$$

Assume that the effective two-level system is initially in an arbitrary state $|A\rangle=\mu|e\rangle+\nu|g\rangle$, and the fields are in a general correlated two-mode state $|F\rangle=\sum_n R_n^q|n+q,n\rangle$. The matrix elements of the reduced field density operator are given by

$$\begin{aligned}\rho_{n,m;q}^F(t)&=\sum_{k=g,e}\langle n,n+q,k|\hat{U}(t)\hat{\rho}(0)\hat{U}^\dagger(t)|k,m+q,m\rangle\\ &=|\mu|^2[U_{ge}(n-1,q)U_{eg}^\dagger(m-1,q)R_{n-1}^q(R_{m-1}^q)^*+U_{ee}(n,q)U_{ee}^\dagger(m,q)R_n^q(R_m^q)^*]\\ &\quad+\mu\nu^*[U_{ge}(n-1,q)U_{gg}^\dagger(m-1,q)R_{n-1}^q(R_m^q)^*+U_{ee}(n,q)U_{ge}^\dagger(m,q)R_n^q(R_{m+1}^q)^*]\\ &\quad+\mu^*\nu[U_{gg}(n-1,q)U_{eg}^\dagger(m-1,q)R_n^q(R_{m-1}^q)^*+U_{eg}(n,q)U_{ee}^\dagger(m,q)R_{n+1}^q(R_m^q)^*]\\ &\quad+|\nu|^2[U_{gg}(n-1,q)U_{gg}^\dagger(m-1,q)R_n^q(R_m^q)^*+U_{eg}(n,q)U_{ge}^\dagger(m,q)R_{n+1}^q(R_{m+1}^q)^*].\end{aligned} \tag{3.6}$$

In the following calculations, for simplicity, we assume that the atom is in the excited state initially. For the pair coherent state with $z=|z|\exp(i\phi)$, the coefficient R_n^q can be written as $R_n^q=f_n^q\exp(in\phi)$, where $f_n^q=\mathcal{N}_q|z|^n/[n!(n+q)!]^{1/2}$. Thus, Eq. (3.6) is reduced to

$$\rho^F_{n,m;q}(t)=[U_{ge}(n-1,q)U^\dagger_{eg}(m-1,q)f^q_{n-1}f^q_{m-1}+U_{ee}(n,q)U^\dagger_{ee}(m,q)f^q_nf^q_m]e^{i(n-m)\phi} . \tag{3.7}$$

IV. PHASE DYNAMICS OF THE TWO-MODE FIELDS

In the Pegg-Barnett phase formalism [13,14], the phase states $|\theta_m\rangle$ of a quantized mode are introduced, which are defined on a finite $(s+1)$-dimensional subspace Ψ spanned by the photon number states $|0\rangle,|1\rangle,\ldots,|s\rangle$,

$$|\theta_m\rangle=(s+1)^{-1/2}\sum_{n=0}^{s}\exp(in\theta_m)|n\rangle , \tag{4.1}$$

where $\theta_m=\theta_0+2\pi m/(s+1)$ $(m=0,1,\ldots,s)$, and the reference phase θ_0 is an arbitrary real number. Accordingly, the Hermitian phase operator is defined as

$$\hat{\Phi}=\sum_{m=0}^{s}\theta_m|\theta_m\rangle\langle\theta_m| . \tag{4.2}$$

Note that the parameter s is taken to be infinity after all expectation values are calculated.

Using Eqs. (4.1) and (4.2), the unitary operator $\exp(i\hat{\Phi})$ can be constructed as a cycling operator in Ψ, which is described as

$$\exp(i\hat{\Phi})\equiv|0\rangle\langle 1|+|1\rangle\langle 2|+\cdots|s-1\rangle\langle s| +\exp[i(s+1)\theta_0]|s\rangle\langle 0| . \tag{4.3}$$

For any physically realizable state, it is easy to show that the average of the unitary operator $\exp(im\hat{\Phi})$ can be written as

$$\langle\exp(im\hat{\Phi})\rangle=\left\langle\sum_{n=0}^{\infty}|n\rangle\langle n+m|\right\rangle . \tag{4.4}$$

In analogy with Eqs. (4.1) and (4.2), the associated phase states and phase operators for the correlated two-mode fields can be generalized straightforwardly by [16]

$$\begin{aligned}|\theta_{j_1},\theta_{j_2}\rangle&=[(s+q+1)(s+1)]^{-1/2}\sum_{n=0}^{s}\exp[i(n+q)\theta_{j_1}+in\theta_{j_2}]|n+q,n\rangle\\&=[(s+q+1)(s+1)]^{-1/2}\exp(iq\theta_{j_1})\sum_{n=0}^{s}\exp[in(\theta_{j_1}+\theta_{j_2})]|n+q,n\rangle ,\end{aligned} \tag{4.5}$$

and

$$\hat{\Phi}_i=\sum_{j_1=0}^{s}\sum_{j_2=0}^{s}\theta_{j_i}|\theta_{j_1},\theta_{j_2}\rangle\langle\theta_{j_2},\theta_{j_1}| , \tag{4.6}$$

where $\theta_{j_i}=\theta_0^{(i)}+2\pi j_i/(s+1), j_i=0,1,\ldots,s$, and $i=1,2$. Therefore, the joint probability distribution $P(\theta_{j_1},\theta_{j_2})$ is given by

$$\begin{aligned}P(\theta_{j_1},\theta_{j_2})&=\mathrm{Tr}(\hat{\rho}^F|\theta_{j_1},\theta_{j_2}\rangle\langle\theta_{j_2},\theta_{j_1}|)\\&=[(s+q+1)(s+1)]^{-1}\sum_{n,m=0}^{s}\rho^F_{n,m;q}(t)e^{i(m-n)\theta_{j_1}}e^{i(m-n)\theta_{j_2}}\\&=[(s+q+1)(s+1)]^{-1}\sum_{n,m=0}^{s}Z(n,m,q;t)f^q_nf^q_me^{i(m-n)(\theta_{j_1}+\theta_{j_2}-\phi)} ,\end{aligned} \tag{4.7}$$

where

$$Z(n,m,q;t)=U_{ge}(n,q;t)U^\dagger_{eg}(m,q;t) +U_{ee}(n,q;t)U^\dagger_{ee}(m,q;t) .$$

In the continuous limit, i.e., $\theta_{j_i}\to\theta_i$ and $2\pi/(s+1)\to d\theta_i$, the joint probability distribution becomes

$$P(\theta_1,\theta_2)=\frac{1}{4\pi^2}\sum_{n,m=0}^{\infty}Z(n,m,q;t)f^q_nf^q_me^{i(m-n)(\theta_1+\theta_2-\phi)} . \tag{4.8}$$

To be specific, we let $\theta_0^{(1)}=\theta_0^{(2)}=\phi/2-\pi$, so that the two phases are symmetrized with respect to $\phi/2$. It is easy to verify that the joint probability distribution equation (4.8) is normalized, i.e.,

$$\int_{\phi/2-\pi}^{\phi/2+\pi}\int_{\phi/2-\pi}^{\phi/2+\pi}P(\theta_1,\theta_2)d\theta_1d\theta_2=1 . \tag{4.9}$$

Obviously, the joint probability distribution equation (4.8) depends explicitly on the sum of the two phases, i.e., $P(\theta_1,\theta_2)=P(\theta=\theta_1+\theta_2)$. This interesting property results from the strong correlation of the two modes, which makes the pair coherent states a partial phase state, and leads to the uniformity of the marginal phase distributions,

$$P(\theta_1)=\int_{\phi/2-\pi}^{\phi/2+\pi} P(\theta_1,\theta_2)d\theta_2=\frac{1}{2\pi}=P(\theta_2) . \quad (4.10)$$

According to Eq. (4.10), we have

$$\langle \hat{\Phi}_1 \rangle=\int_{\phi/2-\pi}^{\phi/2+\pi} \theta_1 P(\theta_1)d\theta_1=\frac{\phi}{2}=\langle \hat{\Phi}_2 \rangle \quad (4.11)$$

and

$$\langle \hat{\Phi}_1^2 \rangle=\int_{\phi/2-\pi}^{\phi/2+\pi} \theta_1^2 P(\theta_1)d\theta_1=\frac{\phi^2}{4}+\frac{\pi^2}{3}=\langle \hat{\Phi}_2^2 \rangle . \quad (4.12)$$

Consequently,

$$\langle \hat{\Phi}_1-\hat{\Phi}_2 \rangle=0 , \quad \langle \hat{\Phi}_1+\hat{\Phi}_2 \rangle=\phi \quad (4.13)$$

and

$$\langle (\Delta\hat{\Phi}_1)^2 \rangle=\langle (\Delta\hat{\Phi}_2)^2 \rangle=\frac{\pi^2}{3} . \quad (4.14)$$

From Eqs. (4.11) and (4.14), it is clear that the individual phases θ_1 and θ_2 remain uniformly distributed as time evolves. Moreover, the mean values of the phase operators are not affected by the Stark shifts, this is quite different from the results of Ref. [25] in which the initial fields are assumed as the direct product of two uncorrelated coherent states. In this paper, instead of the two individual phases θ_1 and θ_2, we examine the fluctuations of the sum of the two phases, $\theta=\theta_1+\theta_2$. Let $\hat{\Phi}_s=\hat{\Phi}_1+\hat{\Phi}_2$, i.e., the phase-sum operator, and thus the variance of $\hat{\Phi}_s$ is given by

$$\langle (\Delta\hat{\Phi}_s)^2 \rangle=\langle (\Delta\hat{\Phi}_1)^2 \rangle+\langle (\Delta\hat{\Phi}_2)^2 \rangle+2C_{\theta_1\theta_2} , \quad (4.15)$$

where

$$\begin{aligned} C_{\theta_1\theta_2}&=\langle \hat{\Phi}_1\hat{\Phi}_2 \rangle-\langle \hat{\Phi}_1 \rangle\langle \hat{\Phi}_2 \rangle \\ &=\int_{\phi/2-\pi}^{\phi/2+\pi}\int_{\phi/2-\pi}^{\phi/2+\pi} \theta_1\theta_2 P(\theta_1,\theta_2)d\theta_1 d\theta_2-\frac{\phi^2}{4} \\ &=-2\,\mathrm{Re}\sum_{m>n} Z(n,m,q;t)\frac{f_n^q f_m^q}{(n-m)^2} \end{aligned} \quad (4.16)$$

is the correlation between the two phases. It is seen that the variance of the phase-sum operator $\hat{\Phi}_s$ as well as the phase-difference operator, $\hat{\Phi}_d=\hat{\Phi}_1-\hat{\Phi}_2$, is determined entirely by the correlation coefficient $C_{\theta_1\theta_2}$. Since $Z(n,m,q;0)=1$ implying a negative correlation coefficient, the variance of $\hat{\Phi}_s$ ($\hat{\Phi}_d$) is smaller (greater) than the sum of the two individual phase variances. For $|z|\to\infty$ at $t=0, C_{\theta_1\theta_2}$ approaches $-\pi^2/3$, indicating that $\langle (\Delta\hat{\Phi}_s)^2 \rangle\to 0$ and $\langle (\Delta\hat{\Phi}_d)^2 \rangle\to 4\pi^2/3$. Thus, in the absence of atom-field interaction, the phase-sum operator $\hat{\Phi}_s$ is a well-defined phase sum in the classical limit, and is therefore considered as the physically relevent variable in this paper.

In general, the joint probability distribution equation (4.8) cannot be expressed as a simple closed form for arbitrary q and r. However, for the special case in which $q=0$ and the Stark shifts are neglected, the joint probability distribution $P(\theta_1,\theta_2)$ can be obtained exactly. In this case, we have

$$\begin{aligned} Z(n,m,q;t)&=\cos[g(n+1)t]\cos[g(m+1)t]+\sin[g(n+1)t]\sin[g(m+1)t] \\ &=\cos[g(n-m)t] \end{aligned} \quad (4.17)$$

and

$$f_n^0=[I_0(2|z|)]^{-1/2} . \quad (4.18)$$

Thus,

$$\begin{aligned} P(\theta=\theta_1+\theta_2)&=\frac{1}{4\pi^2}\left|\sum_{n=0}^{\infty}\cos[g(n+1)t]e^{in(\phi-\theta)}f_n^0\right|^2+\frac{1}{4\pi^2}\left|\sum_{n=0}^{\infty}\sin[q(n+1)t]e^{in(\phi-\theta)}f_n^0\right|^2 \\ &=\frac{1}{8\pi^2 I_0(2|z|)}[e^{2|z|\cos(gt+\phi-\theta)}+e^{2|z|\cos(gt-\phi+\theta)}] . \end{aligned} \quad (4.19)$$

Using the generating function for the Bessel functions $I_n(x)$, the variance of the operator $\hat{\Phi}_s$ can be expressed as

$$\langle (\Delta\hat{\Phi}_s)^2 \rangle=\frac{2\pi^2}{3}-\frac{4}{I_0(2|z|)}\sum_{n=1}^{\infty}\frac{I_n(2|z|)}{n^2}\cos(ngt) . \quad (4.20)$$

Apparently, the variance $\langle (\Delta\hat{\Phi}_s)^2 \rangle$ shows periodic temporal behavior with a period $T=2\pi/g$. The minima and the maxima of the variance are

$$\begin{aligned} \langle (\Delta\hat{\Phi}_s)^2 \rangle_{\min}&=\frac{2\pi^2}{3}-\frac{4}{I_0(2|z|)}\sum_{n=1}^{\infty}\frac{I_n(2|z|)}{n^2} \quad \text{at } t=2l\pi/g , \\ \langle (\Delta\hat{\Phi}_s)^2 \rangle_{\max}&=\frac{2\pi^2}{3}-\frac{4}{I_0(2|z|)}\sum_{n=1}^{\infty}\frac{I_n(2|z|)(-1)^n}{n^2} \quad \text{at } t=(2l+1)\pi/g , \end{aligned} \quad (4.21)$$

where $l=0,1,2,\ldots$. For the asymptotically large $|z|$,

$$I_n(2|z|)\simeq\frac{\exp(2|z|)}{\sqrt{4\pi|z|}}\left[1-\frac{4n^2-1}{16|z|}\right] . \tag{4.22}$$

Hence, we have

$$\lim_{|z|\to\infty}\langle(\Delta\hat{\Phi}_s)^2\rangle_{\min}=0 , \quad \lim_{|z|\to\infty}\langle(\Delta\hat{\Phi}_s)^2\rangle_{\max}=\pi^2 . \tag{4.23}$$

In what follows, we derive the approximate form of Eq. (4.8) for generally q and r. In the strong-field limit, for $n\sim|z|\gg|q|$, the coefficient f_n^q can be well approximated by

$$f_n^q=\mathcal{N}_q\frac{|z|^n}{\sqrt{n!(n+q)!}}=\mathcal{N}_q\frac{|z|^n}{n!\sqrt{(n+1)\cdots(n+q)}}\simeq\mathcal{N}_q\frac{|z|^n}{n!|z|^{q/2}}=\frac{|z|^n}{n!\sqrt{I_q(2|z|)}} , \tag{4.24}$$

which has been verified in fair agreement with the exact one by numerical calculations. Let $a=r^{-1}-r$ and $b=r^{-1}+rq$, such that the detuning can be expressed as

$$D_n^q=g\left[\frac{n}{r}-r(n+q+1)\right]=g\left[(n+1)a-b\right] . \tag{4.25}$$

Consequently, the Rabi frequency is given by

$$\begin{aligned}Q_n^q&=[\tfrac{1}{4}(D_n^q)^2+g^2(n+q+1)(n+1)]^{1/2}\\&=g\left[\frac{(r^{-1}+r)^2}{4}(n+1)^2+\left[q-\frac{ab}{2}\right](n+1)+\frac{b^2}{4}\right]^{1/2} .\end{aligned} \tag{4.26}$$

For $n\gg1$, if r is finite, we have

$$Q_n^q\simeq g[\alpha(n+1)+\beta]\equiv\chi_n^q , \tag{4.27}$$

where $\alpha=(r^{-1}+r)/2$ and $\beta=b/2$.

Next, we examine the evolution term in Eq. (4.7), which can be decomposed into the real and imaginary parts, i.e.,

$$Z(n,m,q;t)=X(n,m,q;t)+iY(n,m,q;t) , \tag{4.28}$$

where

$$X(n,m,q;t)=\cos(Q_n^qt)\cos(Q_m^qt)+\frac{\sin(Q_n^qt)\sin(Q_m^qt)}{Q_n^qQ_m^q}\left[g^2\sqrt{(n+q+1)(n+1)(m+q+1)(m+1)}+\frac{D_n^qD_m^q}{4}\right]$$

and

$$Y(n,m,q;t)=\frac{D_m^q}{2Q_m^q}\sin(Q_m^qt)\cos(Q_n^qt)-\frac{D_n^q}{2Q_n^q}\sin(Q_n^qt)\cos(Q_m^qt) .$$

For $n\gg1$, the following two approximations are feasible:

$$\frac{g\sqrt{(n+q+1)(n+1)}}{Q_n^q}\simeq\frac{n+1+q/2}{\alpha(n+1)+\beta}\simeq\frac{1}{\alpha} \tag{4.29}$$

and

$$\frac{D_n^q}{Q_n^q}\simeq\frac{(n+1)a-b}{\alpha(n+1)+\beta}=2\frac{(1-r^2)(n+1)-1-r^2q}{(1+r^2)(n+1)+1+r^2q}\simeq2\epsilon , \tag{4.30}$$

where $\epsilon=(1-r^2)/(1+r^2)$.

Using Eqs. (4.24)–(4.30), the approximate form for $P(\theta_1,\theta_2)$ in the strong fields is obtained, which can be written as

$$P(\theta=\theta_1+\theta_2)\simeq\frac{1}{4\pi^2 I_q(2|z|)}\sum_{n,m=0}^{\infty}\{\cos(\chi_n^q t)\cos(\chi_m^q t)+\sin(\chi_n^q t)\sin(\chi_m^q t)[\alpha^{-2}+\epsilon^2]$$
$$+i\epsilon[\cos(\chi_n^q t)\sin(\chi_m^q t)-\cos(\chi_m^q t)\sin(\chi_n^q t)]\}e^{i(n-m)(\phi-\theta)}\frac{|z|^{m+n}}{n!m!}$$
$$=\frac{1}{8\pi^2 I_q(2|z|)}[(1-\epsilon)e^{2|z|\cos(\alpha gt+\phi-\theta)}+(1+\epsilon)e^{2|z|\cos(\alpha gt-\phi+\theta)}]\ . \tag{4.31}$$

Consequently, the phase variance is given by

$$\langle(\Delta\hat{\Phi}_s)^2\rangle=\frac{2\pi^2 I_0(2|z|)}{3I_q(2|z|)}-\frac{4}{I_q(2|z|)}\sum_{n=1}^{\infty}\frac{I_n(2|z|)}{n^2}\cos(n\alpha gt)$$
$$\simeq\frac{2\pi^2}{3}\left[1+\frac{q^2}{4|z|}\right]-4\sum_{n=1}^{\infty}\frac{\cos(n\alpha gt)}{n^2}\left[1+\frac{q^2-n^2}{4|z|}\right]\ . \tag{4.32}$$

As a result of the strong-field approximations, both the phase probability distribution and the phase variance exhibit periodic temporal behavior. Using Eq. (4.30), the variance of $\hat{\Phi}_s$ is shown in Fig. 1 for the pair coherent state with $|z|=100$ and $q=2$. It is found that the phase variance shows exactly periodic oscillations with a period $T=2\pi/\alpha g$. We notice that, although the results in Fig. 1 are illustrated in accordance with Eq. (4.32), it has been shown that these numerical results are in fair agreement (in an error of order $|z|^{-1}$) with those obtained by the exact expressions (4.15) and (4.16). Equations (4.31) and (4.32) reduce to the exact expressions (4.19) and (4.20), respectively, in the case of $r=1$ and $q=0$. It seems that if the two levels $|g\rangle$ and $|e\rangle$ are equally coupled ($r=1$), then the Stark effects can be significantly eliminated in the strong fields. According to our previous results [24], the Stark effects are completely eliminated when $r=1$ and $q=-1$.

Although we have derived the periodic phase properties in the strong-field approximations, however, it is interesting to point out that similar periodic behavior is also observed in the presence of small fields. The variance of $\hat{\Phi}_s$ for the pair coherent state with $|z|=5$ and $q=2$ is shown in Fig. 2, which are obtained via Eqs. (4.15) and (4.16). We see that the variance exhibit almost the same behavior as plotted in Fig. 1, except the extreme values which are determined by the mean photon numbers of the fields. In order to compare with the atomic Rabi oscillations, in Fig. 3 we show the probability of finding the atom in the excited state. The phenomena of collapses and revivals (note that the stationary values of the excitation probability are shifted due to the Stark effects) are found in the Rabi oscillations, which appear around the characteristic time $t=l\pi/\alpha g$, where $l=0,1,2,\ldots$. However, as the Rabi oscillations are not periodic, these two oscillations are different in nature. Basically, the atomic inversion are determined only by the diagonal elements of the reduced field density operator. While in determining the phase properties of the cavity fields, the off-diagonal elements of the reduced field density operator are involved, which lead to the delicate cancellation of q in the oscillation frequencies.

We proceed to examine the properties of $\cos\hat{\Phi}_s$ and $\sin\hat{\Phi}_s$. Following Eq. (4.4), the expectation value of $\exp(im\hat{\Phi}_s)$ for the pair coherent state can be generalized as [16]

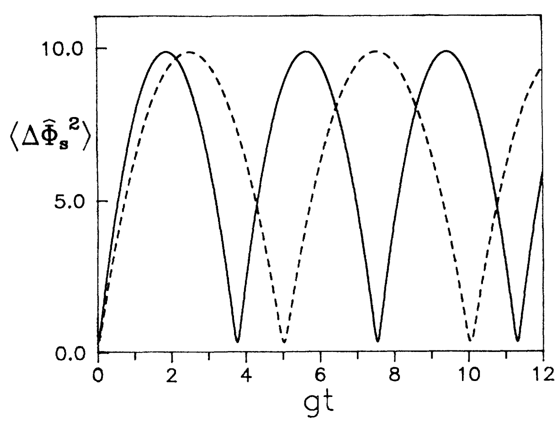

FIG. 1. The variance of the phase operator $\hat{\Phi}_s$ as a function of time. The pair coherent state is prepared with $|z|=100, q=2$. The dashed line is for the case of $r=2$, and the solid line is for the case of $r=3$.

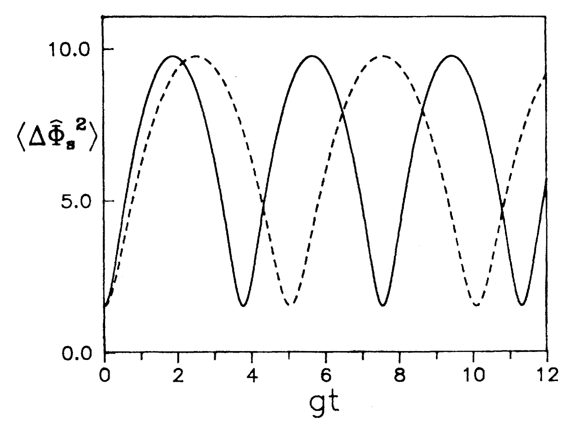

FIG. 2. The same as Fig. 1, except $|z|=5$.

$$\begin{aligned}\langle\exp(im\hat{\Phi}_s)\rangle&=\langle\exp[im(\hat{\Phi}_1+\hat{\Phi}_2)]\rangle=\langle\exp(im\hat{\Phi}_1)\exp(im\hat{\Phi}_2)\rangle\\&=\mathrm{Tr}(\hat{\rho}^F|n+q,n\rangle\langle n+m,n+q+m|)=\sum_{n=0}^{\infty}\rho^F_{n+m,n;q}(t)\\&=e^{im\phi}\sum_{n=0}^{\infty}Z(n+m,n,q;t)f^q_{n+m}f^q_n\ .\end{aligned}\tag{4.33}$$

Thus, the averages of $\cos(m\hat{\Phi}_s)$ and $\sin(m\hat{\Phi}_s)$ are given by

$$\begin{aligned}\langle\cos(m\hat{\Phi}_s)\rangle&=\sum_{n=0}^{\infty}[\cos(m\phi)X(n+m,n,q;t)-\sin(m\phi)Y(n+m,n,q;t)]f^q_{n+m}f^q_n\\&=\mathcal{N}_q^2|z|^m\cos(m\phi)\sum_{n=0}^{\infty}\frac{|z|^{2n}X(n+m,n,q;t)}{\sqrt{n!(n+q)!(n+m)!(n+q+m)!}}\\&\quad-\mathcal{N}_q^2|z|^m\sin(m\phi)\sum_{n=0}^{\infty}\frac{|z|^{2n}Y(n+m,n,q;t)}{\sqrt{n!(n+q)!(n+m)!(n+q+m)!}}\ ,\end{aligned}\tag{4.34}$$

$$\begin{aligned}\langle\sin(m\hat{\Phi}_s)\rangle&=\sum_{n=0}^{\infty}[\cos(m\phi)Y(n+m,n,q;t)+\sin(m\phi)X(n+m,n,q;t)]f^q_{n+m}f^q_n\\&=\mathcal{N}_q^2|z|^m\cos(m\phi)\sum_{n=0}^{\infty}\frac{|z|^{2n}Y(n+m,n,q;t)}{\sqrt{n!(n+q)!(n+m)!(n+q+m)!}}\\&\quad+\mathcal{N}_q^2|z|^m\sin(m\phi)\sum_{n=0}^{\infty}\frac{|z|^{2n}X(n+m,n,q;t)}{\sqrt{n!(n+q)!(n+m)!(n+q+m)!}}\ .\end{aligned}\tag{4.35}$$

In the strong-field approximations, Eqs. (4.24)–(4.30), we have

$$\begin{bmatrix}\langle\cos(m\hat{\Phi}_s)\rangle\\\langle\sin(m\hat{\Phi}_s)\rangle\end{bmatrix}\simeq\frac{I_m(2|z|)}{I_q(2|z|)}\begin{bmatrix}\cos(m\alpha gt)&\epsilon\sin(m\alpha gt)\\-\epsilon\sin(m\alpha gt)&\cos(m\alpha gt)\end{bmatrix}\begin{bmatrix}\cos(m\phi)\\\sin(m\phi)\end{bmatrix}.\tag{4.36}$$

Consequently, the variances of $\cos\hat{\Phi}_s$ and $\sin\hat{\Phi}_s$ are approximated by

$$\begin{aligned}\langle(\Delta\cos\hat{\Phi}_s)^2\rangle&=\langle\cos^2\hat{\Phi}_s\rangle-\langle\cos\Phi_s\rangle^2=\tfrac{1}{2}+\tfrac{1}{2}\langle\cos(2\hat{\Phi}_s)\rangle-\langle\cos\hat{\Phi}_s\rangle^2\\&\simeq\frac{1}{2}+\left[\frac{I_2(2|z|)}{2I_q(2|z|)}-\frac{1+\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\right]\cos(2\phi)\cos(2\alpha gt)\\&\quad+\frac{\epsilon}{2}\left[\frac{I_2(2|z|)}{I_q(2|z|)}-\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\right]\sin(2\phi)\sin(2\alpha gt)\\&\quad-\frac{1-\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2[\cos(2\phi)+\cos(2\alpha gt)]-\frac{1+\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\end{aligned}\tag{4.37}$$

and

$$\begin{aligned}\langle(\Delta\sin\hat{\Phi}_s)^2\rangle&=\langle\sin^2\hat{\Phi}_s\rangle-\langle\sin\hat{\Phi}_s\rangle^2=\tfrac{1}{2}-\tfrac{1}{2}\langle\cos(2\hat{\Phi}_s)\rangle-\langle\sin\hat{\Phi}_s\rangle^2\\&\simeq\tfrac{1}{2}-\left[\frac{I_2(2|z|)}{2I_q(2|z|)}-\frac{1+\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\right]\cos(2\phi)\cos(2\alpha gt)\\&\quad-\frac{\epsilon}{2}\left[\frac{I_2(2|z|)}{I_q(2|z|)}-\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\right]\sin(2\phi)\sin(2\alpha gt)\\&\quad+\frac{1-\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2[\cos(2\phi)-\cos(2\alpha gt)]-\frac{1+\epsilon^2}{4}\left[\frac{I_1(2|z|)}{I_q(2|z|)}\right]^2\ .\end{aligned}\tag{4.38}$$

It is seen that in the presence of strong fields, the mean value and the variance of $\cos\hat{\Phi}_s$ (or $\sin\hat{\Phi}_s$) are oscillating periodically with the period $T=2\pi/\alpha g,\pi/\alpha g$, respectively. In the limit of $|z|\to\infty$, Eqs. (4.37) and (4.38) reduce to

$$\langle(\Delta\cos\hat{\Phi}_s)^2\rangle=\frac{1-\epsilon^2}{4}[1-\cos(2\phi)][1-\cos(2\alpha gt)]\ ,\tag{4.39}$$

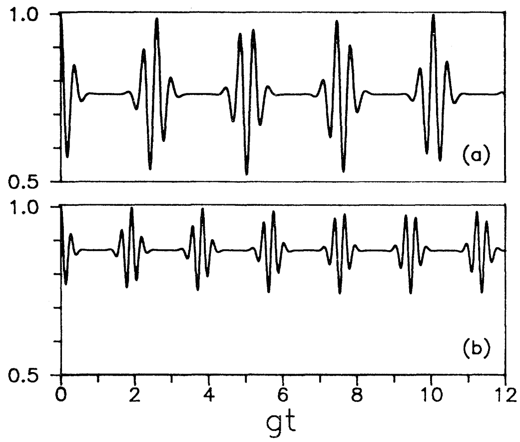

FIG. 3. The probability of finding the atom in the excited state. The pair coherent state is prepared with $|z|=5, q=2$. (a) $r=2$; (b) $r=3$.

$$\langle(\Delta \sin\hat{\Phi}_s)^2\rangle = \frac{1-\epsilon^2}{4}[1+\cos(2\phi)][1-\cos(2\alpha g t)] , \tag{4.40}$$

and it is easy to show that

$$0 \le \begin{cases} \langle(\Delta \cos\hat{\Phi}_s)^2\rangle \\ \langle(\Delta \sin\hat{\Phi}_s)^2\rangle \end{cases} \le 1-\epsilon^2 . \tag{4.41}$$

Thus, for particular ϕ, the variance of $\cos\hat{\Phi}_s$ (or $\sin\hat{\Phi}_s$) always vanishes in the classical limit, and we have

$$\begin{aligned} &\langle(\Delta \cos\hat{\Phi}_s)^2\rangle = 0 , \quad \text{for } \phi = l\pi , \\ &\langle(\Delta \sin\hat{\Phi}_s)^2\rangle = 0 , \quad \text{for } \phi = (l+\tfrac{1}{2})\pi , \end{aligned} \tag{4.42}$$

where l is an integer. We also find that in addition to the choice of particular ϕ, the oscillation amplitudes in Eqs. (4.39) and (4.40) are greatly diminished when the two dipole constants g_1 and g_2 differ considerably in magnitudes.

Finally, according to Eqs. (4.39) and (4.40), we obtain the relations

$$\langle(\Delta \cos\hat{\Phi}_s)^2\rangle + \langle(\Delta \sin\hat{\Phi}_s)^2\rangle = \frac{1-\epsilon^2}{2}[1-\cos(2\alpha g t)] \tag{4.43}$$

and

$$\langle \Delta \cos\hat{\Phi}_s \rangle \langle \Delta \sin\hat{\Phi}_s \rangle = \frac{1-\epsilon^2}{4}\sin(2\phi)[1-\cos(2\alpha g t)] . \tag{4.44}$$

V. CONCLUDING REMARKS

Using the Pegg-Barnett Hermitian phase formalism, we have studied the phase properties for the pair coherent states interacting with the two-mode JCM in which the stark shifts are included. By virtue of the strong correlation of photon numbers in modes, the nontrivial quantity to describe the phase properties for the pair coherent states is the phase-sum operator $\hat{\Phi}_s$. In this paper, we have shown the general expressions for the variance of $\hat{\Phi}_s$ and the variance of $\cos\hat{\Phi}_s$ and $\sin\hat{\Phi}_s$. Furthermore, in the strong-field approximations, the analytic forms for these properties are presented and found to exhibit characteristic oscillations. The periods of these oscillations are characterized by the factor $\alpha = (r^{-1} + r)/2$ which is determined solely by the Stark shifts.

It is seen that the properties of the two individual phases are not affected by the atom-fields interaction. We have shown that the individual phases θ_1 and θ_2 are both uniformly distributed despite the presence of frequency shifts caused by the Stark effects. This interesting property is reflected by the appearance of the parameter α. We see that α is symmetric with respect to r (the ratio of two dipole constants) and its reciprocal, i.e., α is invariant under the exchange $g_1 \leftrightarrow g_2$. This symmetry indicates that the two dipole transitions $|g\rangle \rightarrow i\rangle$ and $|i\rangle \rightarrow |e\rangle$ play indistinguishable roles in determining the dynamical phase properties of pair coherent states in the two-mode JCM, and thus no peculiarity of the individual phases can be singled out.

We have found the characteristic oscillations of phase properties for pair coherent states interacting with an inverted two-level atom. However, for the correlated SU(1,1) coherent states, which is generated with the SU(1,1) analog of the displacement operator in accordance with the definition of Perelomov [26], the characteristic oscillations caused by the Stark shifts are also observed [27]. It seems that the existence of the characteristic oscillations in phase properties is owing to the interaction of the correlated two-mode states with two-level atom. Studies on the phase properties of the correlated SU(1,1) coherent states in the two-mode JCM are planned to be reported elsewhere.

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