行政院國家科學委員會專題研究計畫成果報告

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*   **Hurwitz-Radon 矩陣的分類研究**   *
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共同主持人：

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Classification of Hurwitz-Radon Matrices

Shu-Chu Chang[1] and Cheng Lee[2]
Department of Mathematics
National Changhua University of Education
Changhua, Taiwan, R.O.C 50058

Yichao Xu
Institute of Mathematics, Academia Sinica,
Beijing, China 100080

Abstract

We extend the vector cross-product to a mapping from $F^s \times F^n$ to F^m, where $F = R$ or C. We derive the generalized Hurwitz matrix equation and an equivalence relation of the generalized Hurwitz-Radon matrices from the matrix representation of the mapping with respect to orthonormal bases. Then we use the basic matrix techniques to classify the Hurwitz-Radon matrices and the total invariants under this equivalence relation.

Key words: Hurwitz-Radon matrices, total invariants, vector cross product, canonical form.

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Introduction

B. Eckmann [1] introduced the Hurwitz-Radon matrices; he considered a system of linear transformations $y = A_i x$ $(i = 0, 1, \cdots, s)$ on R^n which have the property

$$\langle A_i x, A_j x\rangle = \delta_{ij} \quad 0 \le i, j \le s$$

or equivalently fulfill the Hurwitz matrix equation

$$A_i' A_j + A_j' A_i = 2\delta_{ij} E$$

where A_i' is the transpose of A_i and E is the $n \times n$ identity matrix. Such matrices $A_0, A_1, \cdots, A_s$ are called Hurwitz-Radon matrices. The original problem considered by Hurwitz [3] and Radon [4] around 1920 concerns the " composition of quadratic forms"

$$\left(\sum_{j=0}^{s} x_j^2\right)\left(\sum_{\ell=1}^{n} y_\ell^2\right) = \sum_{\ell=1}^{n} z_\ell^2,$$

where the z_ℓ are complex bilinear forms of $x_0, \cdots, x_s$ and $y_1, \cdots, y_n$. They determined for given n the maximum number $s+1$ for which such bilinear forms exist.

Hurwitz-Radon Theorem: *If* $n = n_0 \cdot 16^\alpha \cdot 2^\beta$, $\beta = 0, 1, 2, 3$, n_0 *is odd, then* $s_{\max} = 8\alpha + 2^\beta - 1$.

However we prove that $s_{max} = 8\alpha + 2\beta + 1$, which is better than the known result.

Eckmann[1] also introduced a generalization of the vector cross-product in R^n: a bilinear product $\alpha \times \beta\,(\alpha, \beta \in R^n)$ satisfying the norm product rule $|\alpha \times \beta|^2 = |\alpha|^2 |\beta|^2$, but not necessarily the commutative and associative laws.

An equivalent formulation of the norm product rule is

$$\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{j=1}^{n} y_j^2\right) = \sum_{\ell=1}^{n} z_\ell^2$$

where z_ℓ are bilinear of x_i and y_j.

In 1898 Hurwitz proved that such a " composition of quadratic forms" having bilinear functions z_ℓ of x_i and y_j, with real or complex coefficients, can exist for $n = 1, 2, 4, 8$ only. The classical examples for $n = 1, 2, 4$ and 8 are

$$\begin{aligned} R^1 &= R \\ R^2 &= C \\ R^4 &= \text{quaternion algebra } H \\ R^8 &= \text{Cayley numbers or Octonion algebra.} \end{aligned}$$

In this paper we generalize the vector cross-product to a bilinear mapping from $F^s \times F^n$ to F^m ($F : R$ or C), which satisfies the norm product rule. Then we derive the generalized Hurwitz matrix equation. Since the choice of the bases is irrelevant to the mapping, there is an equivalence relation between the matrix representations of the mapping with respect to different sets of orthonormal bases. Eckmann[1] used group representation theory to find canonical forms of the Hurwitz-Radon matrices. In constrast, we employ basic matrix techniques to give not only a complete classification of the Hurwitz-Radon matrices but also their total invariants, and a complete classification of the generalized Hurwitz-Radon matrices for a special case. For this and related subjects, see a recent survey[2].

§1.Generalized Vector Cross-Product and Hurwitz-Radon Matrices.

Let s, n, m be positive integers. We define the *generalized vector cross-product* as a mapping from $C^s \times C^n$ to C^m which fulfill the conditions

$$(a\alpha + \tilde{a}\tilde{\alpha}) \times \beta = a(\alpha \times \beta) + \tilde{a}(\tilde{\alpha} \times \beta) \tag{1}$$
$$\alpha \times (b\beta + \tilde{b}\tilde{\beta}) = \bar{b}(\alpha \times \beta) + \bar{\tilde{b}}(\alpha \times \tilde{\beta})$$

and the norm product rule

$$|\alpha \times \beta| = |\alpha||\beta| \tag{2}$$

for all $a, \tilde{a}, b, \tilde{b} \in C$, $\alpha, \tilde{\alpha} \in C^s$, $\beta, \tilde{\beta} \in C^n$

Here, the commutative and associative laws are not required.

Suppose that with respect to the standard orthonormal bases, the coordinate vectors of α, β and $\alpha \times \beta$ are $x = (x_1, \cdots, x_s)$, $y = (y_1, \cdots, y_n)$ and $z = (z_1, \cdots, z_m)$, respectively.

From (1) we have

$$z_i = f_i(x, y) = xQ_iy^* \qquad i = 1, 2, \cdots, m,$$

where Q_i are $s \times n$ complex matrices, and $y^* = \bar{y}'$ the conjugate transpose of y.

Condition (2) gives rise to "the composition of quadratic forms"

$$|z_1|^2 + \cdots + |z_m|^2 = (|x_1|^2 + \cdots + |x_s|^2)(|y_1|^2 + \cdots + |y_n|^2) \tag{3}$$

Writing (3) in the matrix form, we have

$$\sum_{i=1}^{m} |xQ_iy^*|^2 = x\left(\sum_{i=1}^{m} Q_iy^*yQ_i^*\right)x^* = (xx^*)(yy^*),$$

Since x is arbitrary, we have

$$\sum_{i=1}^{m} Q_i y^* y Q_i^* = yy^* E \qquad \forall y \in C^n,$$

where E (or E_s if the sizes is to be emphasized) is the $s \times s$ identity matrix and Q_i^* is the conjugate transpose of Q_i, or equivalently

$$\sum_{j,k=1}^{n} \bar{y}_j y_k \sum_{i=1}^{m} Q_i e_j' e_k Q_i^* = \left(\sum_{j=1}^{n} y_j \bar{y}_j\right) E,$$

where e_j, e_k are the standard basis of C^n.

Hence we have

$$\sum_{i=1}^{m} Q_i (e_j' e_k + e_k' e_j) Q_i^* = 2\delta_{jk} E, \tag{4}$$

where δ_{jk} is the Kronecker symbol.

Put

$$A_j = \sum_{i=1}^{m} e_i' e_j Q_i^* \qquad (j = 1, 2, \cdots, n), \tag{5}$$

where $e_i \in C^m$, $e_j \in C^n$ (the standard bases).

Then (4) yields

$$A_j^* A_k + A_k^* A_j = 2\delta_{jk} E \qquad (1 \le j, k \le n) \tag{6}$$

This equation is called *the generalized complex Hurwitz matrix equation*, and the complex $m \times s$ matrices $A_1, \cdots, A_n$ are called the *generalized Hurwitz-Radon matrices* if they fulfill equation (6). In particular, when $s = m$ the $m \times m$ matrices $A_1, \cdots, A_n$ are (complex) *Hurwitz-Radon matrices.*

Now, $e_i A_j = e_j Q_i^*$, $A_j^* e_i' = Q_i e_j'$ and it is clear that

$$A_j^* e_i' e_k = Q_i e_j' e_k \qquad \forall k$$

Since x is arbitrary, we have

$$\sum_{i=1}^{m} Q_i y^* y Q_i^* = yy^* E \qquad \forall y \in C^n,$$

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Hence we have

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Now, $e_i A_j = e_j Q_i^*$, $A_j^* e_i' = Q_i e_j'$ and it is clear that

$$A_j^* e_i' e_k = Q_i e_j' e_k \qquad \forall k$$

Thus

$$Q_i = \sum_{k=1}^{n} A_k^* e_i' e_k \qquad (1 \le i \le m) \tag{7}$$

Substituting (7) for Q_i in (4), we can get (6). So (4) and (6) are equivalent.

In the above discussion, we see that generalized Hurwitz-Radon matrices can be derived from the generalized vector cross-product. Conversely, given generalized Hurwitz-Radon matrices $A_1, \cdots, A_n$, put $Q_i = \sum_{j=1}^{n} A_j^* e_i' e_j$ $(1 \le i \le m)$ and $z_i = xQ_iy^*$ $(\forall x \in C^s, y \in C^n)$, we can obtain vectors α, β and γ, whose coordinates are x, y and z, respectively, fulfilling $\gamma = \alpha \times \beta$.

Next we discuss the change of matrix forms of the generalized vector cross-product under different orthonormal bases.

Let U, V, W be unitary change-of-coordinate matrices of size s, n and m, respectively. Let $x, \tilde{x} = xV$ be coordinate vectors of $\alpha \in C^s$, analogously, y, $\tilde{y} = yW$ and z, $\tilde{z} = zU$ be coordinate vectors of $\beta \in C^n$ and $\gamma = \alpha \times \beta \in C^m$, respectively.

Suppose that $z = (z_1, \cdots, z_m)$ with $z_i = xQ_iy^*$ and $\tilde{z} = (\tilde{z}_1, \cdots, \tilde{z}_m)$ with $\tilde{z}_i = \tilde{x}P_i\tilde{y}^*$. Then from $\tilde{z} = zU$ we have

$$\begin{aligned} (xQ_1y^*, \cdots, xQ_my^*)U = \tilde{z} &= (\tilde{x}P_1\tilde{y}^*, ts, \tilde{x}P_m\tilde{y}^*) \\ &= (xVP_1W^*y^*, \cdots, xVP_mW^*y^*) \end{aligned}$$

so

$$xVP_jW^*y^* = (xQ_1y^*, \cdots, xQ_my^*)U^{(j)} = \sum_{i=1}^{m} (xQ_iy^*) e_i U e_j'$$

Therefore, the relationship between P_i and Q_j is

$$P_j = \sum_{i=1}^{m} (e_i U e_j') V^* Q_i W \quad (1 \le j \le m). \tag{8}$$

Suppose that $A_j = \sum_{i=1}^{m} e_i' e_j Q_i^*$ and $B_j = \sum_{i=1}^{m} e_i' e_j P_i^*$. Then

$$Q_i = \sum_{k=1}^{n} A_k^* e_i' e_k \quad \text{and} \quad P_j = \sum_{\ell=1}^{n} B_\ell^* e_j' e_\ell. \tag{9}$$

From (8) and (9) we have

$$\begin{aligned} P_j &= \sum_{i=1}^{m} (e_i U e_j') V^* (\sum_{k=1}^{n} A_k^* e_i' e_k) W \\ &= \sum_{k=1}^{n} V^* A_k^* \sum_{i=1}^{m} e_i' (e_i U e_j') e_k W \\ &= \sum_{k=1}^{n} V^* A_k^* U e_j' e_k W \end{aligned}$$

Hence

$$\begin{aligned} P_j e_t' &= \sum_{k=1}^{n} V^* A_k^* U e_j' e_k W e_t' \\ &= \sum_{k=1}^{n} (e_k W e_t') V^* A_k^* U e_j' \end{aligned}$$

On the other hand, $P_j e_t' = \sum_{\ell=1}^{n} B_\ell^* e_j' e_\ell e_t' = B_t^* e_j'$. Thus

$$B_t = \sum_{k=1}^{n} (e_t W^* e_k') U^* A_k V, \tag{10}$$

which shows the change of the generalized Hurwitz-Radon matrices derived from the generalized vector cross-product with respect to different orthonormal bases.

§2. Complex Hurwitz-Radon Matrices

Given $m \times s$ Hurwitz-Radon matrices $A_1, \cdots, A_n$. Then as shown above these matrices give rise to a generalized vector cross-product $C^s \times C^n \longrightarrow C^m : (\alpha, \beta) \mapsto \alpha \times \beta$, and conversely under some orthonormal bases of C^s, C^n and C^m this generalized vector cross-product can produce another set of generalized Hurwitz-Radon matrices $B_1, \cdots, B_n$. $A_1, \cdots, A_n$ and $B_1, \cdots, B_n$ have the relation (10). Now we would like to find a set of orthonormal bases such that every summand $U^* A_k V$ in (10) has as simple a form as possible. In this section we shall do this for the case of $s = m$.

Note that equation

$$A_j A_k^* + A_k A_j^* = 2\delta_{jk} E$$

is just an equivalent formulation of (6). Obviously A_j are unitary. Firstly, we consider the equivalence relation

$$A_j \mapsto U^* A_j V \qquad j = 1, 2, \cdots, n \tag{11}$$

with U and V unitary. Without loss of generality, we may assume that $A_1 = E$ then $U^* A_1 V = A_1$ implies that $U = V$. Take $k = 1$ and $j \geq 2$ we see that $A_j + A_j^* = 0$; that is, A_j are unitary skew-Hermitian matrices. Hence there is an unitary matrix U such that

$$U^* A_2 U = \sqrt{-1} \begin{pmatrix} E_r & \\ & -E_{m-r} \end{pmatrix}$$

Now, we may assume that

$$A_1 = E$$

$$A_2 = \sqrt{-1}\begin{pmatrix} E_r & \\ & -E_{m-r} \end{pmatrix}$$

For $j \geq 3$, we have $A_2^* A_j + A_j^* A_2 = 0$. Partition A_j into 4 blocks with (1,1) block of size r. A simple calculation shows that $A_j = \begin{pmatrix} 0 & C_j \\ -C_j^* & 0 \end{pmatrix}$ and C_j $(j = 3, \cdots, n)$ fulfill

$$\begin{cases} C_j C_k^* + C_k C_j^* = 2\delta_{jk} E_r \\ C_j^* C_k + C_k^* C_j = 2\delta_{jk} E_{m-r} \end{cases} \quad 3 \leq j, k \leq n$$

When $j = k$ we have $C_j C_j^* = E_r$, so $r \leq m - r$; $C_j^* C_j = E_{m-r}$, so $m - r \leq r$. Therefore $m = 2r$ is even.

On the other hand, if unitary matrix U fulfills $U^* A_1 U = A_1$ and $U^* A_2 U = A_2$ then U=diag(U_1, U_2), where U_1 and U_2 are both unitary and of size $\frac{m}{2}$. So the equations $U^* A_j U = A_j$ for $j \geq 3$ yield $U_1^* C_j U_2 = C_j$ for $j \geq 3$. In other words, the similarity relation $A_j \mapsto U^* A_j U$ reduces to $C_j \mapsto U_1^* C_j U_2$, where U_1, U_2 are of size $\frac{m}{2}$. Continuing the same process, we can show that

$$m = 2^{[\frac{n-1}{2}]} M \quad \text{for some} \quad M \in N$$

and A_j's have the canonical forms $R_1, \cdots, R_n$ as given below.

Theorem 1. *Suppose that $A_1, \cdots, A_n$ are complex $m \times m$ Hurwitz-Radon matrices. Then*
(i) there exists a positive integer M such that

$$m = 2^{[\frac{n-1}{2}]} M,$$

where $[x]$ *denotes the Gauss integer of* x,

(ii) there exist unitary matrices U *and* V *such that* $U^*A_jV = R_j$ $(j = 1, \cdots, n)$, *where* $R_1 = E$

$$R_{2j} = \sqrt{-1}\begin{pmatrix} & & \begin{pmatrix} E_{\frac{m}{2^j}} & 0 \\ 0 & -E_{\frac{m}{2^j}} \end{pmatrix} \\ & \cdot^{\cdot^{\cdot}} & \\ \begin{pmatrix} E_{\frac{m}{2^j}} & 0 \\ 0 & -E_{\frac{m}{2^j}} \end{pmatrix} & & \end{pmatrix}$$

$$R_{2j+1} = \begin{pmatrix} & & \begin{pmatrix} 0 & E_{\frac{m}{2^j}} \\ -E_{\frac{m}{2^j}} & 0 \end{pmatrix} \\ & \cdot^{\cdot^{\cdot}} & \\ \begin{pmatrix} 0 & E_{\frac{m}{2^j}} \\ -E_{\frac{m}{2^j}} & 0 \end{pmatrix} & & \end{pmatrix}$$

for $j = 1, 2, \cdots, [\frac{n-1}{2}]$.

(iii) when n *is even,*

$$R_n = \sqrt{-1}\begin{pmatrix} & & R(p) \\ & \cdot^{\cdot^{\cdot}} & \\ R(p) & & \end{pmatrix}$$

where $R(p) = diag(\underbrace{1, \cdots, 1}_{p}, \underbrace{-1, \cdots, -1}_{M-p})$ *for* $0 \leq p \leq M$.

(iv) if n *is odd, the set of canonical forms* $\{R_1, \cdots, R_n\}$ *is unique;*

(v) if n is even, there are $M+1$ sets of canonical forms, and the total invariants under the equivalence relation (11) is p, where $0 \leq p \leq M$.

Proof: What are left to show is (v).

Suppose that $R_1, \cdots, R_{n-1}, R_n$ and $\tilde{R}_1 = R_1, \cdots, \tilde{R}_{n-1} = R_{n-1}, \tilde{R}_n$ are two sets of canonical forms, where R_n and $\tilde{R}_n$ are determined by nonnegative integers p and $\tilde{p}$, respectively. If there are unitary matrices U and V such that

$$U_0^* R_i V_0 = \tilde{R}_i \qquad i = 1, 2, \cdots, n$$

A direct calculation reveals

$$R_0 = R_1 R_2^* R_3 R_4^* \cdots R_{n-1} R_n^* = (-\sqrt{-1})^{\frac{n}{2}} \mathrm{diag}(R(p), \cdots, R(p))$$
$$\tilde{R}_0 = \tilde{R}_1 \tilde{R}_2^* \tilde{R}_3 \tilde{R}_4^* \cdots \tilde{R}_{n-1} \tilde{R}_n^* = (-\sqrt{-1})^{\frac{n}{2}} \mathrm{diag}(R(\tilde{p}), \cdots, R(\tilde{p})).$$

Because $\tilde{R}_0 = U_0^* R_0 U_0$,

$$\begin{aligned} \mathrm{diag}(R(\tilde{p}), \cdots, R(\tilde{p})) &= U_0^* \mathrm{diag}(R(p), \cdots, R(p)) U_0 \\ &= U_0^{-1} \mathrm{diag}(R(p), \cdots, R(p)) U_0. \end{aligned}$$

That is to say diag $(R(\tilde{p}), \cdots, R(\tilde{p}))$ and diag $(R(p), \cdots, R(p))$ are similar. Hence they have the same number of positive eigenvalue 1, so $\tilde{p} = p$, and $\tilde{R}_i = R_i$ $(1 \leq i \leq n)$. This shows that, under the equivalence relation (11), the total invariants are nonnegative integers $p = 0, 1, \cdots, M$. The proof is complete.

Next, consider the canonical forms and the total invariants of the complex Hurwitz-Radon matrices $A_1, \cdots, A_n$ under the equivalence relation (10), we have

Theorem 2. *The canonical forms of A_i $(i = 1, 2, \cdots, n)$ under the equivalence relation (10) are still $R_1, \cdots, R_n$ given in Theorem 2.1. Moreover, they are unique when n is odd; otherwise, the total invariants are $p \in \{[\frac{M+1}{2}], \cdots, M\}$ and in fact if two sets $R_1, \cdots, R_n$ and $\tilde{R}_1 = R_1, \cdots, \tilde{R}_{n-1} = R_{n-1}, \tilde{R}_n$ are equivalent under (10) then $\tilde{p} = p$ or $\tilde{p} = M - p$.*

Proof. Suppose there are unitary matrices W, U, V such that

$$\tilde{R}_j = \sum_{i=1}^{n}(e_j W^* e_i')U^* R_i V \qquad 1 \le j \le n$$

Then

$$\begin{aligned} & \tilde{R}_1 \tilde{R}_2^* \tilde{R}_3 \cdots \tilde{R}_{n-1} \tilde{R}_n^* \\ = & \sum_{i_1, i_2, \cdots i_n = 1}^{n} (e_1 W^* e_{i_1}')(e_{i_2} W e_2') \cdots (e_{n-1} W^* e_{i_{n-1}}')(e_{i_n} W e_n') U^* R_{i_1} R_{i_2}^* \cdots R_{i_n}^* U \end{aligned}$$

If $i_j = i_{j+1}$ then $R_{i_j} R_{i_{j+1}}^* = E = R_{i_j}^* R_{i_{j+1}}$ and the coefficients which involve with $i_j = i_{j+1} \in \{1, 2, \cdots, n\}$ are 0, for

$$\sum_{i_j = i_{j+1} = 1}^{n} (e_j W^* e_{i_j}')(e_{i_j} W e_{j+1}') = e_j e_{j+1}' = 0.$$

Hence

$\tilde{R}_1 \tilde{R}_2^* \cdots \tilde{R}_{n-1} \tilde{R}_n^*$

$= \sum_{i_1, \cdots, i_n} (e_1 W^* e_{i_1}')(e_{i_2} W e_2') \cdots (e_{i_n} W e_n') U^* R_{i_1} R_{i_2}^* \cdots R_{i_n}^* U,$

where $i_1, i_2, \cdots, i_n$ are pairwise distinct.

Since $A_i^* A_j = -A_j^* A_i$ and $A_i A_j^* = -A_j A_i^*$ for $i \ne j$, we have $R_i^* R_j = -R_j^* R_i$ and $R_i R_j^* = -R_j R_i^*$ for $i \ne j$. Therefore

$$\tilde{R}_1 \tilde{R}_2^* \cdots \tilde{R}_{n-1} \tilde{R}_n^* = \triangle U^* R_1 R_2^* \cdots R_n^* U$$

where $\Delta = \sum \text{sign}(i_1 i_2 \cdots i_n)\ (e_1 W^* e'_{i_1})(e_{i_2} W e'_2) \cdots (e_{i_n} W e'_n)$, the summation runs over all permutations of S_n and

$$\text{sign}(i_1 i_2 \cdots i_n) = \begin{cases} 1 & \text{if } (i_1 \cdots i_n) \text{ is an even permutation} \\ -1 & \text{otherwise} \end{cases}$$

Clearly $\Delta \in C$. As in the proof of Theorem 2.1, we have

$$\text{diag}(R(\tilde{p}), \cdots, R(\tilde{p})) = \Delta U^* \text{diag}(R(p), \cdots, R(p)) U$$

On the left-hand side, there are $2^{[\frac{n-1}{2}]}$ blocks of $R(\tilde{p})$, hence the number of eigenvalue 1 is $2^{[\frac{n-1}{2}]}\tilde{p}$, and the rest of the diagonal entries are -1. Similarly, there are $2^{[\frac{n-1}{2}]}p$ many Δ and $2^{[\frac{n-1}{2}]}(M-p)$ many $-\Delta$ in the diagonal matrix of the right-hand side. Therefore we have shown that either (1) $\Delta = 1$ and $\tilde{p} = p$ or (2) $\Delta = -1$ and $\tilde{p} = M - p$. In order to give a complete set of canonical forms, we may choose $p \geq M - p$ or $p \geq \frac{M}{2}$; that is, pick $p \in \{[\frac{M+1}{2}], ..., M\}$. We complete the proof.

§3. Real Hurwitz-Radon matrices

In this section we give a complete classification of the real Hurwitz-Radon matrices. Let $A_1, ..., A_n$ be real $m \times m$ Hurwitz-Radon matrices, fulfilling

$$A'_i A_j + A'_j A_i = 2\delta_{ij} E \qquad 1 \leq i, j \leq n \tag{12}$$

As in theorem 1, from the fact that A_j $(j \geq 2)$ are skew-symmetric and orthogonal, we may show easily that $2 \mid m$, and there exist real orthogonal matrix U such that $U' A_2 U = \begin{pmatrix} 0 & -E_{\frac{m}{2}} \\ E_{\frac{m}{2}} & 0 \end{pmatrix}$.

Now, assume that $A_1 = E$, $A_2 = \begin{pmatrix} 0 & -E_{\frac{m}{2}} \\ E_{\frac{m}{2}} & 0 \end{pmatrix}$ and partition A_j $(j \geq 3)$ into 4 square blocks, say $A_j = \begin{pmatrix} X_j & Y_j \\ U_j & V_j \end{pmatrix}$. From $A_j' A_i + A_i' A_j = 0$ $(i = 1, 2)$, we conclude that $X_j' = -X_j = V_j$ and $Y_j' = -Y_j = U_j' = -U_j$; i.e., $A_j = \begin{pmatrix} X_j & Y_j \\ Y_j & -X_j \end{pmatrix}$. Set $Z_j = X_j + \sqrt{-1} Y_j$ then $Z_j' = -Z_j$, hence Z_j is complex skew-symmetric, and (12) is equivalent to

$$\begin{cases} X_j X_k + X_k X_j + Y_j Y_k + Y_k Y_j = -2\delta_{jk} E_{\frac{m}{2}} \\ X_j Y_k - Y_j X_k + X_k Y_j - Y_k X_j = 0 \end{cases}$$

This brings on the Hurwitz matrix equation

$$Z_j^* Z_k + Z_k^* Z_j = 2\delta_{jk} E_{\frac{m}{2}} \quad j, k \geq 3.$$

Since the orthogonal matrix U fulfills

$$U' \begin{pmatrix} 0 & -E_{\frac{m}{2}} \\ E_{\frac{m}{2}} & 0 \end{pmatrix} U = \begin{pmatrix} 0 & -E_{\frac{m}{2}} \\ E_{\frac{m}{2}} & 0 \end{pmatrix},$$

U has the form $\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$, where P, Q satisfy

$$(P + \sqrt{-1} Q)^* = (P + \sqrt{-1} Q)^{-1} = P' - \sqrt{-1} Q'.$$

Moreover, let $\tilde{A}_j = U' A_j U$ with $\tilde{A}_j = \begin{pmatrix} \tilde{X}_j & \tilde{Y}_j \\ \tilde{Y}_j & -\tilde{X}_j \end{pmatrix}$, then

$$\tilde{X}_j = P' X_j P - Q' Y_j Q - P' Y_j Q - Q' X_j Q,$$

and

$$\tilde{Y}_j = P' X_j Q - Q' Y_j Q + P' Y_j P + Q' X_j P.$$

Thus

$$\tilde{Z}_j = \tilde{X}_j + \sqrt{-1}\tilde{Y}_j = (P + \sqrt{-1}Q)' Z_j (P + \sqrt{-1}Q) = W' Z_j W.$$

So we have

Lemma 1. *Let $A_1 = E, A_2, \cdots, A_n$ be $m \times m$ real Hurwitz-Radon matrices. Then*

(i) $2 \mid m$ and there exists an orthogonal matrix U such that

$$U' A_2 U = \begin{pmatrix} 0 & -E_{\frac{m}{2}} \\ E_{\frac{m}{2}} & 0 \end{pmatrix}, U' A_j U = \begin{pmatrix} X_j & Y_j \\ Y_j & -X_j \end{pmatrix} \; for\, j \geq 3;$$

(ii) let $Z_j = X_j + \sqrt{-1}Y_j$ then $Z_j' = -Z_j$ and

$$Z_j^* Z_k + Z_k^* Z_j = 2\delta_{jk} E \quad for\ j, k \geq 3;$$

(iii) if $A_2 = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, then U has the form $\begin{pmatrix} P & Q \\ -Q & P \end{pmatrix}$;
(iv) let $W = P + \sqrt{-1}Q$ then W is unitary and the equivalence relation $A_j \mapsto U' A_j U$ is transformed into $Z_j \mapsto W' Z_j W$.

The following two results, due to Hua, will be needed later. Since there are rare English references to these, we give the proofs here.

Lemma 2. *Let A be $n \times n$ complex matrix, U be unitary. If $UA = AU'$ then there is an unitary matrix V such that $VA = AV'$ and $V^2 = U$.*

Proof: U is normal, there is an unitary matrix U_1 such that

$$U_1^* U U_1 = diag(e^{\sqrt{-1}\theta_1}, \cdots, e^{\sqrt{-1}\theta_n}) = D$$

Put $V = U_1 diag(e^{\frac{\sqrt{-1}\theta_1}{2}}, \cdots, e^{\frac{\sqrt{-1}\theta_n}{2}})U_1^*$, then $V^2 = U$. From $UA = AU'$ we have $U_1 D U_1^* A = A(U_1 D U_1^*)' = A\bar{U}_1 D U_1'$, and $DU_1^* A\bar{U}_1 = U_1^* A \bar{U}_1 D$. If $U^* A\bar{U}_1 = (b_{ij})$ then $(e^{\sqrt{-1}\theta_i} - e^{\sqrt{-1}\theta_j})b_{ij} = 0$, so $(e^{\frac{\sqrt{-1}\theta_i}{2}} - e^{\frac{\sqrt{-1}\theta_j}{2}})b_{ij} = 0$. Therefore

$$diag(e^{\frac{\sqrt{-1}\theta_1}{2}}, \cdots, e^{\frac{\sqrt{-1}\theta_n}{2}})U_1^* A\bar{U}_1 = U_1^* A\bar{U}_1 diag(e^{\frac{\sqrt{-1}\theta_1}{2}}, \cdots, e^{\frac{\sqrt{-1}\theta_n}{2}})$$

or

$$\begin{aligned} VA &= U_1 diag(e^{\frac{\sqrt{-1}\theta_1}{2}}, \cdots, e^{\frac{\sqrt{-1}\theta_n}{2}})U_1^* A \\ &= A\bar{U}_1 diag(e^{\frac{\sqrt{-1}\theta_1}{2}}, \cdots, e^{\frac{\sqrt{-1}\theta_n}{2}})U_1' = AV'. \end{aligned}$$

Lemma 3. *Let A be $m \times m$ unitary matrix.*
(i) If $A' = A$ then there is an unitary matrix U such that $U'AU = E$.
(ii) If $A' = -A$ then $2 \mid m$ and there is an unitary matrix U such that $U'AU = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$.

Proof: (i) There is an unitary matrix P such that

$$PAP^* = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n) \text{ with } \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0.$$

Clearly $(P'P)A = A(P'P)'$. By lemma 2, there is an unitary matrix V such that $V^2 = P'P$ and $VA = AV'$. Set $U = V'P^* diag(\frac{1}{\sqrt{\lambda_1}}, \cdots, \frac{1}{\sqrt{\lambda_n}})$. Then

$$U'AU = diag(\frac{1}{\sqrt{\lambda_1}}, \cdots, \frac{1}{\sqrt{\lambda_n}})\bar{P}VAV'P^* diag(\frac{1}{\sqrt{\lambda_1}}, \cdots, \frac{1}{\sqrt{\lambda_n}}) = E$$

(ii) Since $det(A) \neq 0$ and $det(A) = det(A') = (-1)^m det(A)$, m must be even. Set $U_1 = diag\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right)$. Then

$$(U_1 A^*)A = A(U_1 A^*)'$$

By lemma 2 there is an unitary matrix V such that $V^2 = U_1 A^*$, $VA = AV'$, and

$$VAV' = U_1 = diag\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \cdots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

Interchanging rows and corresponding columns such that VAV' is unitarily congruent to $\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$. That is, there is an unitary matrix U such that

$$UAU' = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}.$$

Lemma 4. *Let $T_1, ..., T_{n_0}$ be $m_1 \times m_1$ complex Hurwitz-Radon matrices and T_i be skew-symmetric. Then there exists a positive integer M such that*

$$m_1 = 2^{[\frac{n_0-1}{2}]} M$$

Moreover, there is a unitary matrix U such that

$$U' T_j U = B_1 R_j \quad for \quad j = 1, 2, \cdots, n_0,$$

where $R_1, \cdots, R_{n_0}$ are the canonical forms given in theorem 2, B_1 is orthogonal and

$$B_\ell = \begin{pmatrix} C_\ell & 0 \\ 0 & -C_\ell \end{pmatrix}, \quad C_\ell = \begin{pmatrix} 0 & -B_{\ell+1} \\ B_{\ell+1} & 0 \end{pmatrix}, \quad \ell = 1, 2, \cdots$$

This is a recursive block-simplifying process of B_1, and there are 8 cases of reaching the final block of the process:

(i) if $n_0 = 8r-3$, the final block unit is $B_{2r} = E$;

(ii) if $n_0 = 8r-2$, the final block unit is $B_{2r} = E$;

(iii) if $n_0 = 8r-1$, the final block unit is $C_{2r} = E$;

(iv) if $n_0 = 8r$, the final block unit is $F_{2r} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, and $p = \frac{M}{2}$

(v) if $n_0 = 8r+1$, the final block unit is $B_{2r+1} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$

(vi) if $n_0 = 8r+2$, the final block unit is

$$B_{2r+1} = \begin{pmatrix} \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}_{p\times p} & \\ & \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}_{(M-p)\times(M-p)} \end{pmatrix}$$

(vii) if $n_0 = 8r+3$, the final block unit is $B_{2r+2} = E$

(viii) if $n_0 = 8r+4$, the final block unit is $B_{2r+2} = E$ and $p = \frac{M}{2}$.

When n_0 is even, the total invariants is p.

Proof: Consider the $m_1 \times m_1$ complex matrices $T_1^*T_1, T_1^*T_2, \cdots, T_1^*T_{n_0}$. Since $T_1, \cdots, T_{n_0}$ are Hurwitz-Radon matrices, we have

$$\begin{cases} T_1^*T_1 = E \\ (T_1^*T_i)^*(T_1^*T_j) + (T_1^*T_j)^*(T_1^*T_i) = 2\delta_{ij}E \end{cases}$$

By theorem 2, there exists positive integer M such that $m_1 = 2^{[\frac{n_0-1}{2}]}M$ and there exists unitary matrix U such that $U^*(T_1^*T_i)U = R_i$, where R_i are given in theorem 2. Under the equivalence relation $T_i \longrightarrow U'T_iU = \tilde{T}_i$, we have

$$\tilde{T_1}^*\tilde{T_j} = (U'T_1U)^*(U'T_jU) = U^*T_1^*\bar{U}U'T_jU = U^*(T_1^*T_j)U = R_j$$

Hence $\tilde{T_j} = \tilde{T_1}R_j$ for $j = 1, 2, \cdots, n$. Write $\tilde{T_1} = B_1$, then we have shown the first part of the theorem.

Next, we determine B_1. So far we have

$$\begin{cases} B_1' = -B_1, \qquad B_1B_1^* = E, \\ B_1R_{2j} = R_{2j}B_1, B_1R_{2j+1} = -R_{2j+1}B_1 \text{ and} \\ B_1R_{2j}R_{2j+1}^* = R_{2j}B_1R_{2j+1}^* = R_{2j}(-B_1R_{2j+1}) = R_{2j}R_{2j+1}^*B_1 \end{cases}$$

Partitioning B_1 into 4 square blocks and from condition $B_1' = -B_1$, $B_1R_2 = R_2B_1$ and $B_1R_3 = -R_3B_1$, we know that B_1 must be of the form $\begin{pmatrix} C_1 & 0 \\ 0 & -C_1 \end{pmatrix}$. Analogously, partitioning C_1 into 4 square blocks and applying $B_1R_4 = R_4B_1$ and $B_1R_5 = -R_5B_1$, we know that C_1 must be of the form $\begin{pmatrix} 0 & -B_2 \\ B_2 & 0 \end{pmatrix}$, and so on. If $k = [\frac{n_0-1}{4}]$ then we have $B_1 = \tilde{T_1}, B_\ell = \begin{pmatrix} C_\ell & 0 \\ 0 & -C_\ell \end{pmatrix}$ and

$$C_\ell = \begin{pmatrix} 0 & -B_{\ell+1} \\ B_{\ell+1} & 0 \end{pmatrix}$$

for $\ell = 1, 2, ..., k$.

Moreover, it is just a routine work to see that

(1) when $n_0 = 4k+1$, the final block unit is B_{k+1};

(2) when $n_0 = 4k+2$, the final block unit is $B_{k+1} = \begin{pmatrix} C_{k+1} & 0 \\ 0 & D_{k+1} \end{pmatrix}$, where C_{k+1} is $p \times p$ and D_{k+1} is $(M-p) \times (M-p)$

(3) when $n_0 = 4k+3$, the final block unit is C_{k+1}.

(4) when $n_0 = 4k+4$, the final block unit is $C_{k+1} = \begin{pmatrix} 0 & \tilde{D}_{k+2} \\ D_{k+2} & 0 \end{pmatrix}$, where $\tilde{D}_{k+2}$ is $p \times (M-p)$, D_{k+2} is $(M-p) \times p$

From condition $B_1 + B_1' = 0$ we have

$C_{2q}' = C_{2q},\ C_{2q+1}' = -C_{2q+1};\ B_{2q}' = B_{2q},\ B_{2q+1}' = -B_{2q+1}$.

From condition $B_1^* B_1 = E$, we have $C_q^* C_q = E$ and $B_q^* B_q = E$. Thus

(i) if $k = 2r-1$ and $n_0 = 4k+1 = 8r-3$, the final block unit is $B_{k+1} = B_{2r}$, where $B_{2r}' = B_{2r}$ and $B_{2r}^* B_{2r} = E$;

(ii) if $k = 2r$ and $n_0 = 4k+1 = 8r+1$, the final block unit is $B_{k+1} = B_{2r+1}$, where $B_{2r+1}' = -B_{2r+1}$ and $B_{2r+1}^* B_{2r+1} = E$;

(iii) if $k = 2r-1$ and $n_0 = 4k+2 = 8r-2$, the final block unit is

$B_{k+1} = B_{2r} = \begin{pmatrix} C_{2r} & 0 \\ 0 & D_{2r} \end{pmatrix}$, where C_{2r} is $p \times p$ and $C_{2r}' = C_{2r},\ C_{2r}^* C_{2r} = E$;

while D_{2r} is $(M-p) \times (M-p)$ and $D_{2r}' = D_{2r},\ D_{2r}^* D_{2r} = E$;

(iv) if $k = 2r$ and $n_0 = 4k+2 = 8r+2$, the final block unit is $B_{k+1} = B_{2r+1} = \begin{pmatrix} C_{2r+1} & 0 \\ 0 & D_{2r+1} \end{pmatrix}$, where $C_{2r+1}' = -C_{2r+1},\ C_{2r+1}^* C_{2r+1} = E,\ D_{2r+1}' = -D_{2r+1}$, $D_{2r+1}^* D_{2r+1} = E$;

(v) if $k = 2r-1$ and $n_0 = 4k+3 = 8r-1$, the final block unit is $C_{k+1} = C_{2r}$, where $C_{2r}' = C_{2r},\ C_{2r}^* C_{2r} = E$;

(vi) if $k = 2r$ and $n_0 = 4k+3 = 8r+3$, the final block unit is $C_{k+1} = C_{2r+1}$, where $C_{2r+1}' = -C_{2r+1},\ C_{2r+1}^* C_{2r+1} = E$;

(vii) if $k = 2r-1$ and $n_0 = 4k+4 = 8r$, the final block unit is $C_{k+1} = C_{2r}$

$$= \begin{pmatrix} 0 & \tilde{D}_{2r+1} \\ D_{2r+1} & 0 \end{pmatrix}, \text{ and } C'_{2r} = C_{2r},\ C^*_{2r}C_{2r} = E;$$

(viii) if $k = 2r$ and $n_0 = 4k + 4 = 8r + 4$, the final block unit is $C_{2r+1} =$

$$\begin{pmatrix} 0 & \tilde{D}_{2r+2} \\ D_{2r+2} & 0 \end{pmatrix}, \text{ where } C'_{2r+1} = -C_{2r+1},\ C^*_{2r+1}C_{2r+1} = E.$$

On the other hand, if an unitary matrix U fulfills $R_j = U^*R_jU$, then $U = diag(U_1, U_1, \cdots, U_1)$ where U_1 is $M \times M$. If n_0 is even, $U_1 = diag(U_{11}, U_{12})$, where U_{11} is unitary of size p and U_{12} is unitary of size $M - p$. This has brought about applying similarity relation, using U_1, to the final block units of the above 8 cases. We complete the proof by lemma 3.

Theorem 3. *Let $A_1, \cdots, A_n$ be $m \times m$ real Hurwitz-Radon matrices . Then there exists positive integer M such that $m = 2^{[\frac{n-1}{2}]}M$. Let*

$$\tilde{A}_i = \sum_{j=1}^{n} (e_i W' e'_j) U' A_j V$$

Then the canonical forms of $\tilde{A}_1, \cdots, \tilde{A}_n$ are $P_1 = E$, $P_2 = \begin{pmatrix} 0 & -E_1 \\ E_1 & 0 \end{pmatrix}$,

$$P_{2j+1} = \begin{pmatrix} B_1R_{2j-1} & 0 \\ 0 & -B_1R_{2j-1} \end{pmatrix},\ P_{2j+2} = \begin{pmatrix} 0 & -\sqrt{-1}B_1R_{2j} \\ -\sqrt{-1}B_1R_{2j} & 0 \end{pmatrix}, \text{ where}$$

(1) $R_1, R_2, \cdots, R_{n-2}$ are $\frac{m}{2} \times \frac{m}{2}$ complex matrices given by theorem 1, and hence $1 \le j \le \frac{n-1}{2}$ if n is odd, $1 \le j \le \frac{n-2}{2}$ if n is even;

(2) $R_{n-2} = \sqrt{-1}\begin{pmatrix} & & R(p) \\ & \cdot & \\ & \cdot & \\ R(p) & & \end{pmatrix}$ *depends on the nonnegative integer* p;

(3) B_1 *is* $\frac{m}{2} \times \frac{m}{2}$ *real matrix , defined by*

$$B_\ell = \begin{pmatrix} C_\ell & 0 \\ 0 & -C_\ell \end{pmatrix}, \qquad C_\ell = \begin{pmatrix} 0 & -B_{\ell+1} \\ B_{\ell+1} & 0 \end{pmatrix} \qquad \ell = 1, 2, \cdots$$

the final block unit is of the form.

(i) $n = 8r - 1$, $B_{2r} = E$;

(ii) $n = 8r$, $B_{2r} = E$, $M \geq p \geq \frac{M}{2}$;

(iii) $n = 8r + 1$, $C_{2r} = E$;

(iv) $n = 8r + 2$, $C_{2r} = \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix}$, $2 \mid M$ *and* $p = \frac{M}{2}$;

(v) $n = 8r + 3$, $B_{2r+1} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$;

(vi) $n = 8r + 4$, $M \geq p \geq \frac{M}{2}$, $2 \mid p$, $2 \mid M$

$$B_{2r+1} = diag(\begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}_{p\times p}, \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}_{(M-p)\times(M-p)});$$

(vii) $n = 8r + 5$, $B_{2r+2} = E$;

(viii) $n = 8r + 6$, $2 \mid M$, $p = \frac{M}{2}$, $B_{2r+2} = E$.

Proof: Take $n_0 = n - 2$ in lemma 4, so $m = 2m_1 = 2 \cdot 2^{[\frac{n-3}{2}]}M = 2^{[\frac{n-1}{2}]}M$. We need only discuss the cases of $n = 8r$ and $8r + 4$, since the canonical forms of the remaining cases are unique.

Let $n = 8r$. If canonical forms $P_1, \cdots, P_n$ and $\tilde{P}_1, \cdots, \tilde{P}_n$ are equivalent, where

$$\tilde{P}_1 = P_1, \tilde{P}_2 = P_2,$$

$$\tilde{P}_{2j+1} = \begin{pmatrix} \tilde{B}_1 \tilde{R}_{2j-1} & 0 \\ 0 & -\tilde{B}_1 \tilde{R}_{2j-1} \end{pmatrix}, \tilde{P}_{2j+2} \begin{pmatrix} 0 & -\sqrt{-1}\tilde{B}_1 \tilde{R}_{2j} \\ -\sqrt{-1}\tilde{B}_1 \tilde{R}_{2j} & 0 \end{pmatrix}$$

for $1 \leq j \leq 4r-1$, and $\tilde{B}_1$, $\tilde{R}_{2j}$ are determined by nonnegative integer $\tilde{p}$. That is, there exist orthogonal matrices W_0, U_0 and V_0 such that

$$\tilde{P}_i = \sum_{j=1}^{n} (e_i W_0' e_j') U_0' P_j V_0, \ 1 \leq i \leq n.$$

Conditions $B_1 R_{2j} = R_{2j} B_1$, $B_1 R_{2j-1} = -R_{2j-1} B_1$, $B_1^{-1} = B_1' = -B_1$, $B_1^2 = -E$ imply $\tilde{B}_1 \tilde{R}_{2j} = \tilde{R}_{2j} B_1$, $\tilde{B}_1 \tilde{R}_{2j-1} = -\tilde{R}_{2j-1} \tilde{B}_1$, $\tilde{B}_1^{-1} = \tilde{B}_1' = -\tilde{B}_1$, $\tilde{B}_1^2 = -E$.

Therefore

$$\tilde{P}_1 \tilde{P}_2' \cdots \tilde{P}_{n-1} \tilde{P}_n' = \sum_{j_1, \cdots, j_n = 1}^{n} (e_1 W_0' e_{j_1}')(e_{j_2} W_0 e_2') \cdots (e_{j_n} W_0 e_n') U_0' P_{j_1} P_{j_2}' \ldots P_{j_n}' U_0$$

As in theorem 2, we have $\tilde{P}_1 \tilde{P}_2' \cdots \tilde{P}_{n-1} \tilde{P}_n' = \Delta U_0' P_1 P_2' \cdots P_n' U_0$.
A direct calculation reveals that

$$(-\sqrt{-1})^{\frac{2-n}{2}} P_1 P_2' \cdots P_{n-1} P_n' = \begin{pmatrix} R_1 R_2^* \ldots R_{n-3} R_{n-2}^* & 0 \\ 0 & R_1 R_2^* \cdots R_{n-3} R_{n-2}^* \end{pmatrix}$$

$$(-\sqrt{-1})^{\frac{2-n}{2}} \tilde{P}_1 \tilde{P}_2' \cdots \tilde{P}_{n-1} \tilde{P}_n' = \begin{pmatrix} R_1 R_2^* \cdots R_{n-3} \tilde{R}_{n-2}^* & 0 \\ 0 & R_1 R_2^* \cdots R_{n-3} \tilde{R}_{n-2}^* \end{pmatrix}$$

This shows that

$$\begin{aligned} &diag(E_{\tilde{p}}, -E_{M-\tilde{p}}, \cdots, E_{\tilde{p}}, -E_{(M-\tilde{p})}) \\ &= \Delta U_0' diag(E_p, -E_{(M-p)}, \cdots, E_p, -E_{(M-p)}) U_0 \end{aligned}$$

Therefore, if $\triangle = 1$ then $\tilde{p} = p$; if $\triangle = -1$ then $\tilde{p} = M - p$.

Conversely, if $\tilde{p} = p$ then $\tilde{P}_j = P_j$ for $1 \leq j \leq n$; if $\tilde{p} = M - p$, take $\tilde{W} = diag(1, \cdots, 1, -1)$. It is not difficult to show that $\{P_1, \cdots, P_n\}$ and $\{\tilde{P}_1, \cdots, \tilde{P}_n\}$ are equivalent. Hence we may always assume that $p \geq \frac{M}{2}$ whenever n is even, and the canonical forms are uniquely determined by p.

A similar argument can be applied to the case of $n = 8r + 4$ □

In considering the composition of quadratic forms

$$(\sum_0^s x_j^2)(\sum_1^n y_l^2) = \sum_1^n z_l^2,$$

Hurwitz and Radon determined that for a given n the maximun possible number of s is

$$s_{max} = 8\alpha + 2^\beta - 1 \text{ if } n = odd \cdot 16^\alpha \cdot 2^\beta \text{ and } \beta = 0, 1, 2, 3.$$

However, we prove the following result, in which our s_{max} is greater than theirs.

Corollary 1. *Given positive integer n, let s_{max} be the largest s such that $A_0, A_1, \cdots, A_s$ be the $n \times n$ Hurwitz-Radon matrices. Then*

$$s_{max} = 8\alpha + 2\beta + 1,$$

whenever $n = 16^\alpha 2^\beta M$, M is odd and $\beta \in \{0, 1, 2, 3\}$.

Proof: By theorem 3, $n = 2^{[\frac{s+1-1}{2}]}M = 2^{[\frac{s}{2}]}M$, $M \in N$. In order to have s_{max}, we take M to be odd. So $[\frac{s_{max}}{2}] = 4\alpha + \beta$. To reach the maximality, take $s_{max} = 2t + 1$, then $t = 4\alpha + \beta$, and thus

$s_{max} = 2(4\alpha + \beta) + 1 = 8\alpha + 2\beta + 1$ □

Hurwitz also proved that the above "composition of quadratic form" exist only when $n = 1, 2, 4, 8$. As an application of theorem 3, we give the explicit canonical forms for the case of $n = m$. For convenience, let

$$J_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, J_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

Theorem 4. *The only possible forms for $n \times n$ Hurwitz-Radon matrices $A_1, A_2, \cdots, A_n$ are*

(i) $n = 1$, $P_1 = 1$;

(ii) $n = 2$, $P_1 = E_2$, $\quad P_2 = J_2$;

(iii) $n = 4$, $P_1 = E_4$, $P_2 = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix} = diag(J_2, -J_2)$,

$$P_3 = \begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix}, P_4 = \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix};$$

(iv) $n = 8$, $P_1 = E_8$, $P_2 = diag(J_2, -J_2, -J_2, J_2)$,

$$P_3 = diag\left(\begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \right),$$

$$P_4 = diag\left(\begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -J_2 \\ -J_2 & 0 \end{pmatrix} \right), P_5 = \begin{pmatrix} 0 & -E_4 \\ E_4 & 0 \end{pmatrix},$$

$$P_6 = \begin{pmatrix} & \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} \\ \begin{pmatrix} J_2 & 0 \\ 0 & J_2 \end{pmatrix} & \end{pmatrix}, P_7 = \begin{pmatrix} & \begin{pmatrix} 0 & -J_1 \\ J_1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -J_1 \\ J_1 & 0 \end{pmatrix} & \end{pmatrix},$$

$$P_8 = \begin{pmatrix} & \begin{pmatrix} 0 & -J_3 \\ J_3 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -J_3 \\ J_3 & 0 \end{pmatrix} & \end{pmatrix}.$$

Proof: $m = n = 2^{[\frac{n-1}{2}]}M$, implies $1 \leq n \leq 8$; indeed, if $n > 8$ then $2^{[\frac{n-1}{2}]} > n$, and it is impossible to find integer M. Moreover, the equality holds only when $n = 1, 2, 4, 8$. From theorem 3, we know that it is true for $n = 1$ and $n = 2$, and for $n = 4$, we have

$T_1 = E_4$, $T_2 = \begin{pmatrix} 0 & -E_2 \\ E_2 & 0 \end{pmatrix}$, $T_3 = \begin{pmatrix} B_1 & 0 \\ 0 & -B_1 \end{pmatrix} = \begin{pmatrix} 0 & -J_2 \\ -J_2 & 0 \end{pmatrix}$,

$T_4 = \begin{pmatrix} 0 & -J_1 \\ -J_1 & 0 \end{pmatrix}$.

Pick $W = diag(1, 1, 1, -1)$, $U = V = \frac{1}{\sqrt{2}} \begin{pmatrix} J_1 & J_3 \\ J_3 & J_1 \end{pmatrix}$ and set

$$P_j = \sum_{i=1}^{4} (e_j W' e_i') U' T_i V.$$

Then $P_1 = U'T_1U$, $P_2 = U'T_2U$, $P_3 = U'T_3U$, $P_4 = -U'T_4U$. Hence the canonical forms are $P_1 = E_4$, $P_2 = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}$, $P_3 = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$, $P_4 = \begin{pmatrix} 0 & J_2 \\ J_2 & 0 \end{pmatrix}$. Finally, let $n = 8$. We have

$$T_1 = E_8, \qquad T_2 = \begin{pmatrix} 0 & -E_4 \\ E_4 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} B_1 & 0 \\ 0 & -B_1 \end{pmatrix}, \qquad T_4 = \begin{pmatrix} 0 & -\sqrt{-1}B_1R_2 \\ -\sqrt{-1}B_1R_2 & 0 \end{pmatrix},$$

$$T_5 = \begin{pmatrix} B_1R_3 & 0 \\ 0 & -B_1R_3 \end{pmatrix}, \quad T_6 = \begin{pmatrix} 0 & -\sqrt{-1}B_1R_4 \\ -\sqrt{-1}B_1R_4 & 0 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} B_1R_5 & 0 \\ 0 & -B_1R_5 \end{pmatrix}, \quad T_8 = \begin{pmatrix} 0 & -\sqrt{-1}B_1R_6 \\ -\sqrt{-1}B_1R_6 & 0 \end{pmatrix}$$

where

$$R_2 = \sqrt{-1}\begin{pmatrix} E_2 & 0 \\ 0 & -E_2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix},$$

$$R_4 = \sqrt{-1}\begin{pmatrix} 0 & J_1 \\ J_1 & 0 \end{pmatrix}, \quad R_5 = \begin{pmatrix} 0 & -J_2 \\ -J_2 & 0 \end{pmatrix},$$

$$R_6 = \sqrt{-1}\begin{pmatrix} 0 & J_3 \\ J_3 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} J_2 & 0 \\ 0 & -J_2 \end{pmatrix}.$$

Thus

$$T_1 = E_8, \; T_2 = \begin{pmatrix} 0 & -E_4 \\ E_4 & 0 \end{pmatrix},$$

$$T_3 = \begin{pmatrix} \begin{pmatrix} J_2 & \\ & -J_2 \end{pmatrix} & \\ & \begin{pmatrix} -J_2 & \\ & J_2 \end{pmatrix} \end{pmatrix},$$

$$T_4 = \begin{pmatrix} & \begin{pmatrix} J_2 & \\ & J_2 \end{pmatrix} \\ \begin{pmatrix} & J_2 \\ J_2 & \end{pmatrix} & \end{pmatrix},$$

$$
T_5 = \begin{pmatrix} \begin{pmatrix} & J_2 \\ J_2 & \end{pmatrix} & \\ & \begin{pmatrix} & -J_2 \\ -J_2 & \end{pmatrix} \end{pmatrix},
$$

$$
T_6 = \begin{pmatrix} & \begin{pmatrix} & J_3 \\ -J_3 & \end{pmatrix} \\ \begin{pmatrix} & J_3 \\ -J_3 & \end{pmatrix} & \end{pmatrix},
$$

$$
T_7 = \begin{pmatrix} \begin{pmatrix} & E_2 \\ -E_2 & \end{pmatrix} & \\ & \begin{pmatrix} & -E_2 \\ E_2 & \end{pmatrix} \end{pmatrix},
$$

$$
T_8 = \begin{pmatrix} & \begin{pmatrix} & -J_1 \\ J_1 & \end{pmatrix} \\ \begin{pmatrix} & -J_1 \\ J_1 & \end{pmatrix} & \end{pmatrix}.
$$

Pick $P_1 = T_1$, $P_2 = T_3$, $P_3 = -T_7$, $P_4 = T_5$, $P_5 = T_2$, $P_6 = T_4$, $P_7 = T_8$, and $P_8 = -T_6$, or equivalently there are $U = V = E_8$ and an orthogonal matrix W such that $P_i = \sum_{j=1}^{8}(e_i W' e_j')U'T_jV$. □

The following two corollaries are easy to derive.

Corollary 2. *If $A_1, \cdots, A_n$ are $n \times n$ real Hurwitz-Radon matrices. Then*

$$
\sum_{j=1}^{n} A_j'(e_u'e_v + e_v'e_u)A_j = \sum_{j=1}^{n} A_j(e_u'e_v + e_v'e_u)A_j' = 2\delta_{uv}E
$$

where $e_u, e_v \in R^n$, $1 \le u, v \le n$.

Corollary 3. *When* $n = 1, 2, 4$, *the canonical forms* $P_1, \cdots, P_n$ *have the property that* $P_i P_j = \sum_{k=1}^{n}(e_k P_i e_j')P_k$, *for* $1 \le i, j \le n$. *But it fails when* $n = 8$.

§4. Generalized Hurwitz-Radon matrices

In this section we discuss the canonical forms of the generalized Hurwitz-Radon matrices for $n = 1, 2$. Let $A_1, \cdots, A_n$ be $m \times s$ complex Hurwitz-Radon matrices, then $A_1^* A_1 = E_s$, hence $m \ge s$ and there are unitary matrices U and V such that

$$U^* A_1 V = \begin{pmatrix} 0_{(m-s)\times s} \\ E_{s\times s} \end{pmatrix}$$

Therefore, when $n = 1$, the canonical form is $\begin{pmatrix} 0 \\ E_s \end{pmatrix}$. When $n = 2$, we may assume that $A_1 = \begin{pmatrix} 0 \\ E_s \end{pmatrix}$, and $A_2 = \begin{pmatrix} L_{(m-s)\times s} \\ K_{s\times s} \end{pmatrix}$. From $A_1^* A_2 + A_2^* A_1 = 0$, we have $K + K^* = 0$; From $A_2^* A_2 = E$, we have $L^* L + K^* K = E$.

Suppose unitary matrices $U_{m\times m}$ and $V_{s\times s}$ satisfy $U^* A_1 V = A_1$, then $U = \begin{pmatrix} U_1 & 0 \\ 0 & V \end{pmatrix}$ and $U^* A_2 V = \begin{pmatrix} U_1^* L V \\ V^* K V \end{pmatrix}$. Since K is skew-symmetric, we may assume that $K = \sqrt{-1}\,diag(O_p, \wedge_q, E_u, -E_v)$, where $p + q + u + v = s$, $\wedge = diag(a_1, ..., a_q)$ with $a_i \ne 0, \pm 1$, and $a_1 \ge a_2 \ge \cdots \ge a_q$.
Since

$$L^* L = E - K^* K = diag(E_p, E - \wedge^2, 0_u, 0_v) \ge 0, \quad a_i^2 \le 1 \text{ for } 1 \le i \le q.$$

Write $\wedge = diag(b_1 E_{q_1}, ..., b_t E_{q_t})$, with $1 > b_1 > b_2 > ... > b_t > -1$ and $b_j \ne 0$. Partition the columns of L in the way that $s = p + q_1 + ... + q_t + u + v$, and write L as

$$L = (L_0, L_1, ..., L_t, \tilde{L}_+, \tilde{L}_-)$$

Then from $L^*L = diag(E_p, E - \wedge^2, 0, 0)$, we see that $\tilde{L}_+ = 0 = \tilde{L}_-$ and $L_i^*L_j = (1 - b_j^2)\delta_{ij}E$, $0 \le i, j \le t$, where $b_0 = 0$. Take unitary matrix $V = diag(V_0, V_1, \cdots, V_t, E_u, E_v)$, then $V^*KV = K$ and

$$U_1^*LV = (U_1^*L_0V,\ U_1^*L_1V_1,\ \cdots,\ U_1^*L_tV_t,\ 0,\ 0),$$

$$L^*L = V^*L^*LV = (U_1^*LV)^*(U_1^*LV),$$

$$(U_1^*L_iV_i)^*(U_1^*L_iV_i) = (1 - b_i^2)E_{q_i}, \quad 0 \le i \le t, \qquad b_0 = 0, \quad q_0 = p.$$

We may find unitary matrices U_1 and V_0 such that $U_1^*L_0V_0 = \begin{pmatrix} 0 \\ E_p \end{pmatrix}$. Assume that $L_0 = \begin{pmatrix} 0 \\ E \end{pmatrix}$, then from $L_0^*L_i = 0$, $(i = 1, 2, \cdots, t)$, we have $L_i = \begin{pmatrix} L_{i_1} \\ 0 \end{pmatrix}$ $1 \le i \le t$, Furthermore, $L_{i_1}^*L_{j_1} = 0$ (if $i \ne j$), $L_{i_1}^*L_{i_1} = (1 - b_i^2)E_{q_i}$ $1 \le i, j \le t$.

Therefore, there are U_1^* and V such that $U_1^*LV = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_p & 0 & 0 & 0 \\ 0 & (E - \wedge^2)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}$

Thus we have the following theorems:

Theorem 5. *Suppose that the $m \times s$ complex matrix A_1 has the property that $A_1^*A_1 = E$, then $m \ge s$ and there are unitary matrices U and V such* *that*

$$U^* A_1 V = \begin{pmatrix} 0 \\ E \end{pmatrix}.$$

Theorem 6. *Suppose that A_1, A_2 are $m \times s$ complex Hurwitz-Radon matrices, then $m \geq s$ and there are unitary matrices U and V such that*

$$U^* A_1 V = \begin{pmatrix} 0 \\ E \end{pmatrix} \quad \text{and} \quad U^* A_2 V = \begin{pmatrix} L \\ K \end{pmatrix}$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_p & 0 & 0 & 0 \\ 0 & (E - \wedge^2)^{\frac{1}{2}} & 0 & 0 \end{pmatrix}, \quad K = \sqrt{-1} \begin{pmatrix} 0_p & & & \\ & \wedge_q & & \\ & & E_u & \\ & & & -E_v \end{pmatrix}$$

Finally, we consider the canonical forms of the generalized Hurwitz-Radon matrices under the equivalence relation (10) $A_k \longrightarrow \sum_{i=1}^{m} (e_i V^* e_k^{'}) W^* A_i U$. We have

Theorem 7. *Suppose that the $m \times s$ complex matrix A_1 has the property that $A_1^* A_1 = E$ then under the equivalence relation (10), the canonical form is unique and is of the form $\begin{pmatrix} 0 \\ E \end{pmatrix}$.*

Theorem 8. *Suppose that A_1, A_2 are $m \times s$ complex Hurwitz-Radon matri-*

ces. Then under the equivalence relation (10), the canonical forms are

$$P_1 = \begin{pmatrix} 0 \\ E \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_p & 0 & 0 & 0 \\ 0 & (E-\wedge^2)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \wedge_q & 0 & 0 \\ 0 & 0 & E_u & 0 \\ 0 & 0 & 0 & -E_v \end{pmatrix}$$

where $\wedge = diag(a_1, \cdots, a_q)$ *with* $0 < |a_i| < 1$, $1 \le i \le q$, *and* $s = p+q+u+v$, $m \ge 2p+2q+u+v$, $u \ge v$. *If* $u > v$, *the canonical form is unique; if* $u = v$, *replacing* $\wedge$ *by* $-\wedge$ *in* P_2, *and obtain two sets of equivalent canonical forms.*

Proof: Suppose that $\tilde{P}_1 = P_1, \tilde{P}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ E_{\tilde{p}} & 0 & 0 & 0 \\ 0 & (E-\wedge_1^2)^{\frac{1}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \wedge_{1\tilde{q}} & 0 & 0 \\ 0 & 0 & E_{\tilde{u}} & 0 \\ 0 & 0 & 0 & -E_{\tilde{v}} \end{pmatrix}$

is another set of canonical forms, and there are unitary matrices $W_{2\times 2}$, $U_{m\times m}$, $V_{s\times s}$ such that

$$\tilde{P}_i = \sum_{j=1}^{2} (e_i W^* e_j') U^* P_j V, \quad \text{for} \quad i = 1, 2$$

Then

$$\tilde{P}_1^* \tilde{P}_2 = \sum_{j_1, j_2 = 1}^{2} (e_{j_1} W e_1')(e_2 W^* e_{j_2}') V^* P_{j_1}^* P_{j_2} V = \lambda V^* P_1^* P_2 V$$

where $0 \neq \lambda \in C$. Now,

$$\begin{pmatrix} 0_{\tilde{p}} & & & \\ & \wedge_{1\tilde{q}} & & \\ & & E_{\tilde{u}} & \\ & & & -E_{\tilde{v}} \end{pmatrix} = \lambda V^* \begin{pmatrix} 0_p & & & \\ & \wedge_q & & \\ & & E_u & \\ & & & -E_v \end{pmatrix} V$$

where $\Lambda = diag(b_1 E_{q_1 \times q_1}, ..., b_t E_{q_t \times q_t})$ with $1 > b_1 > \cdots > b_w > 0 > b_{w+1} > \cdots > b_t > -1$, and $\Lambda_1 = diag(\tilde{b}_1 E_{\tilde{q}_1 \times \tilde{q}_1}, \cdots, \tilde{b}_{\tilde{t}} E_{\tilde{q}_{\tilde{t}} \times \tilde{q}_{\tilde{t}}})$ with $1 > \tilde{b_1} > \cdots > \tilde{b_{\tilde{w}}} > 0 > b_{\tilde{w}+1} > ... > \tilde{b_{\tilde{t}}} > -1$.

Hence $\tilde{p} = p$ and if (u,v)$\neq 0$ then $(\tilde{u}, \tilde{v}) \neq 0$. Moreover, $\lambda = \pm 1$:

(i) if $\lambda = 1, \tilde{u} = u, \tilde{t} = t, \tilde{b}_i = b_i$;

(ii) if $\lambda = -1, \tilde{u} = v, \tilde{v} = u, \tilde{t} = t$; $(\tilde{b_1}, \cdots, \tilde{b_{\tilde{w}}}) = (-b_t, \cdots, -b_{w+1})$,

$$(\tilde{b}_{\tilde{w}+1}, \cdots, \tilde{b_{\tilde{t}}}) = (-b_w, \cdots, -b_1).$$

So we may say that $u \geq v$. If $u > v$ then $\lambda = 1$, the forms are unique; if $u = v$ then $\lambda = 1$ or -1, and correspondingly replace Λ_1 by Λ and obtain the other equivalent canonical forms. □

Especially, for the real case we have

Theorem 9. *Let A_1, A_2 be $m \times s$ real Hurwitz-Radon matrices. Then $m \geq s$ and there are orthogonal matrices $U_{m\times m}$, $V_{s\times s}$ such that*

$$U'A_1V = \begin{pmatrix} 0 \\ E \end{pmatrix}, \quad U'A_2V = \begin{pmatrix} L \\ K \end{pmatrix}$$

where

$$L = \begin{pmatrix} 0 & 0 & 0 \\ E_{p\times p} & 0 & 0 \\ 0 & (E-K_0K_0')^{\frac{1}{2}} & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0_{p\times p} & 0 & 0 & 0 \\ 0 & K_0 & 0 & 0 \\ 0 & 0 & 0 & -E_{q_0\times q_0} \\ 0 & 0 & E_{q_0\times q_0} & 0 \end{pmatrix}$$

where

$$K_0 = diag(b_1 \begin{pmatrix} 0 & -E_{q_1\times q_1} \\ E_{q_1\times q_1} & 0 \end{pmatrix}, \cdots, b_t \begin{pmatrix} 0 & -E_{q_t} \\ E_{q_t} & 0 \end{pmatrix}),$$

$0 < b_1 < \ldots < b_t < 1$ *and*

$$s = p + 2\sum_{j=0}^{t} q_j, \quad m \geq 2p + 2q_0 + 4\sum_{j=1}^{t} q_j.$$

Furthermore, it is unique under the relation (10)

Proof: As in theorem 8, we may partition the columns of L such that $L = (L_0, L_1, \cdots, L_t, 0)$, where L_0 is $(m-s)\times p$, L_i is $(m-s)\times 2q_i$, $(i = 1, \cdots, t)$, $L_i'L_j = 0$ $(i \neq j)$ and $L_i'L_i = (1-b_i^2)E_{2q_i\times 2q_i}$ with $b_0 = 0$ and $0 < b_1 < \ldots < b_t < 1$. Denote the $(m-s)\times s$ matrix

$$Q = \left(\frac{1}{\sqrt{1-b_0^2}} L_0, \cdots, \frac{1}{\sqrt{1-b_t^2}} L_t \right) = (L_0, \cdots, L_t)\frac{1}{\sqrt{E-\wedge^2}},$$

where

$$\wedge = diag(b_0E_p, \cdots, b_tE_{2q_t}).$$

Then $Q'Q = E_s$. Namely, the column vectors of Q are pairwise orthogonal unit vectors, it can be extended to a $(m-s)\times(m-s)$ real orthogonal unit matrix Q_0, such that $Q_0'Q = \begin{pmatrix} E_s \\ 0 \end{pmatrix}$. Therefore

$$L = (L_0, L_1, \cdots, L_t, 0) = (Q\sqrt{E-\wedge^2}, 0) = (Q_0 \begin{pmatrix} E \\ 0 \end{pmatrix} \sqrt{E-\wedge^2}, 0)$$

$$= Q_0 \begin{pmatrix} \sqrt{E - \Lambda^2} & 0 \\ 0 & 0 \end{pmatrix}.$$

By applying a permutation matrix again, we complete the proof. □

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