行政院國家科學委員會專題研究計畫成果報告

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Describe Rings which can be a Finite Union of Proper Subskew Fields

一個可表為有限個真子斜體的聯集的環

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計畫類別：■ 個別型計畫　　□ 整合型計畫

計畫編號：NSC 89-2115-M-018-005

執行期間：88 年 8 月 1 日至 89 年 7 月 31 日

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執行單位：國立彰化師範大學數學系

中華民國 89 年 9 月 26 日

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Abstract

We characterize rings which is a finite union of proper subskew fields. We have proven that such kind of rings must be semi-simple right Artinian and is isomorphic to Z_2^k, the kth ($k \geq 2$) cartesian power of the two element field Z_2.

The motivation of proposing the question of the title came from an interesting paper [?] of C. Lanski. In there, Lanski gave a characterization of those rings which can be a finite union of proper right annihilators. Thoughout this paper let R be an associative ring. Follows from a Lemma of B. H. Neumann [4; Lemma p239], we show that neither a field nor a skew field can be a finite union of its proper subskew fields. This shows that the question is not trivial. Our main result gives a characterization of those rings which can be a finite union of proper subskew fields.

For convenience, we state the Lemma of Neumann.

Lemma 1: *Let the group G be the union of finitely many, let us say n, cosets of subgroups $H_1, H_2, ..., H_n$:*

$$G = \cup_{i=1}^{n} H_i g_i$$

Then at least one subgroup H_i has finite index in G.

Using this result, we can deal with the case of R being a field and a skew field.

Example 2: *Neither a field nor a skew field can be a finite union of proper subskew fields.*

Proof. Let R be a field. Assume by way of contradiction that $R = \cup_{i=1}^{n} R_i$, where R_i is a proper subfield of R for all $i = 1, 2, ..., n$.

Let $R^* = R \backslash \{0\}$ and $R_i^* = R_i \backslash \{0\}$ for all $i = 1, 2, ..., n$. Then by Neumann's lemma, there exists $1 \leq j \leq n$ such that R_j^* is of finite index in

R^*. That is the factor group R^*/R_j^* is of finite order, say k. Given $x \in R^*$, we consider the element xR_j^* of R^*/R_j^*. Since $|R^*/R_j^*| = k$, we have that $(xR_j^*)^k = R_j^*$. Thus, for all $x \in R$ we get that $x^k \in R_j$.

We claim that $| R_j^* | \leq k$. Assume that $| R_j^* | > k$. Pick distinct elements $\lambda_1, \lambda_2, ..., \lambda_{k+1}$ in R_j^*. Take x in $R \backslash R_j$. Then $(x+\lambda_1)R_j^*, (x+\lambda_2)R_j^*, ..., (x+\lambda_{k+1})R_j^*$ are elements in R^*/R_j^*. Since $| R^*/R_j^* | = k$, we see that $(x+\lambda_i)R_j^*$, $i = 1, 2, ..., k+1$, can not be all distinct. We may assume without loss of generality that $(x+\lambda_1)R_j^* = (x+\lambda_2)R_j^*$. This implies that there exist $r, r' \in R_j^*$ such that

$$(x+\lambda_1)r = (x+\lambda_2)r'.$$

Hence

$$x(r - r') = \lambda_2 r' - \lambda_1 r.$$

Since $\lambda_1, \lambda_2, r, r'$ are in R_j^*, so the right hand side of the equation is in R_j^*. If $r - r' = 0$, we get that $\lambda_1 = \lambda_2$. This contradicts to the assumption that $\lambda_1 \neq \lambda_2$. So $r - r' \neq 0$. Since $r - r' \in R_j^*$, $x = (\lambda_2 r' - \lambda_1 r)(r - r')^{-1} \in R_j$, which contradicts to the fact that $x \in R \backslash R_j$. Hence $| R_j^* | \leq k$.

Recall that $R^* = \cup_{i=1}^{k} C_i$, where the C_i are the k distinct cosets of R_j^*. Since all the C_i's are finite, we see that R is a finite field. Thus R^* is a cyclic group. Let $R^* = < r >$, then $r \in R_l$ for some $l \in \{1, 2, ..., n\}$, and hence $< r > \subseteq R_l$. We have $R_l = R$, this lead to a contradiction.

Now if R is a skew field and suppose that $R = \cup_{i=1}^{n} R_i$, where R_i is a proper subskew field of R for all $i = 1, 2, ..., n$. Recall that there exists $j \in \{1, 2, ..., n\}$ such that $| R^*/R_j^* | = k < \infty$, and for all $x \in R$ we have $x^k \in R_j$. In [?] Herstein give a brief proof of Faith's Theorem which asserts that if D is a skew field and $A \neq D$ a proper subring of D. Suppose that for every $x \in D$, $x^{n(x)} \in A$ where $n(x) \in N$ depends on x. Then D is commutative. By this result, we see that R is commutative, hence R is a field. By above, the assumption fails.

What can we say about a ring which is a finite union of its proper subskew fields? Next lemma gives the necessary condition.

Theorem 3: *If R is a finite union of proper subskew fields then R is a semi-simple right Artinian ring.*

Proof. Let $R = \cup_{i=1}^{n} R_i$, where R_i is a subskew field of R for all $i = 1, 2, \cdots, n$. Clearly $R^2 \neq (0)$. Let ρ be a right ideal of R. Then $\rho = \cup_{i=1}^{n}(\rho \cap R_i)$. Since R_i is a skew field, $\rho \cap R_i = (0)$ or R_i. Therefore there exists a subset $S \subseteq \{1, 2, \cdots, n\}$ such that $\rho = \cup_{i \in S} R_i$. Hence R has only finitely many right ideals, which implies that R is right Artinian.

To complete the proof we have to show that R is semi-simple. It is known that $J(R)$ (the Jacobson radical of R) is a nilpotent ideal if R is right Artinian. Let $x \in J(R)$, then x is a nilpotent element, say $x^m = 0$ for some positive integer m. Then $x = 0$ since it must be in some skew field R_j. Hence $J(R) = 0$. Thus R is semi-simple.

Thus if a ring satisfying the assumption, then it must contains a unity since it is semi-simple right Artinian. Applying the Wedderburn and Artin's theorem we see that R is isomorphic to

$$M_{n_1}(D_1) \oplus ... \oplus M_{n_k}(D_k),$$

where the D_i are skew fields and $M_{n_i}(D_i)$ is the ring of all $n_i \times n_i$ matrices over the skew field D_i for all $i = 1, 2, ..., k$. Hence, the rings satisfying the conditions of the title are those of the form:

$$M_{n_1}(D_1) \oplus ... \oplus M_{n_k}(D_k).$$

Clearly $M_{n_i}(D_i)$ have nilpotent elements if $n_i > 1$. Since R has no nonzero nilpotent elements, we see that all $n_i = 1$, and we have

Corollary 5: *If R is a finite union of proper subskew fields then R is isomorphic to*

$$D_1 \oplus D_2 \oplus \cdots \oplus D_k$$

where the D_i are skew fields for all $i = 1, 2, ..., k$.

By Example 2, we see that $k \geq 2$. Suppose that $R = \cup_{i=1}^{n} R_i$ and n is minimal; (*i.e.*, $\cup_{i \neq j}^{n} R_i \neq R$ for all $j = 1, 2, ..., n,$), then $n \geq k$. We may assume that all $D_i \neq 0$. Then

$$\cup_{i=1}^{n}(R_i \cap D_1) = (\cup_{i=1}^{n} R_i) \cap D_1 = R \cap D_1 = D_1.$$

Choose i such that $R_i \cap D_1 \neq 0$. Since D_1 is an ideal of R, we have that $R_i \cap D_1$ is an ideal of R_i. Therefore

$$R_i = R_i \cap D_1 \subseteq D_1.$$

Hence D_1 is a finite ($\leq n$) union of R_i with $R_i \cap D_1 \neq 0$. Since a skew field can not be a finite union of subskew fields, we have that there exists an i such that $D_1 = R_i$. For simplicity we may assume that $i = 1$; *i.e.*, $D_1 = R_1$. If $R_j \cap D_1 \neq 0$, then $R_j \subseteq D_1$ by the above observation. Therefore $R_j \subseteq R_1$, contradicting to the minimality of n. Similarly, we may assume that $D_j = R_j$ for all $j \leq k$. Moreover, the minimality of n implies that $R_k \cap D_j \neq 0$ if and only if $k = j$.

Let e_j be the unity of D_j for all $j = 1, 2, \cdots, k$. If $a^2 = a$ for all a in D_j, then D_j is a Boolean ring and so $D_j = Z_2$. Suppose that $D_j \neq Z_2$ for some $j \geq 2$, pick a in D_j with $a \neq a^2$. We have $e_1 + a \in R_i$ for some $i \in \{1, 2, \cdots, n\}$. Since $D_1 D_j = (0)$ we see that $(e_1 + a)^2 = e_1 + a^2 \in R_i$. Hence

$$0 \neq a - a^2 = (e_1 + a) - (e_1 + a^2) \in R_i \cap D_j,$$

forcing that $R_i = D_j$ and so $e_1 \in D_j$. This leads to a contradiction. Hence all elements $a \in D_j$ have the property that $a^2 = a$. Thus $D_j = Z_2$. Analogously $D_1 = Z_2$.

Recall that $R = \cup_{i=1}^{n} R_i = Z_2^k$ where $k \geq 2$. We see that R has exactly 2^k elements since $R = Z_2^k$. We claim that $n = 2^k - 1$. Clearly $a^2 = a$ for all $a \in Z_2^k$ and $< a > \cong Z_2$. Let F be a subskew field of Z_2^k. Then F is a subfield and every element of F is a root of $x^2 - x$. This implies that $|F| = 2$ and $F \cong Z_2$. Thus we see that all $R_i \cong Z_2$ for all $i = 1, 2, \cdots, n$. Since R has 2^k elements and each R_i has exactly one nonzero element, we see that $n = 2^k - 1$.

Therefore, we have our main result.

Theorem 5: *Let R be a ring. If $R = \cup_{i=1}^{n} R_i$ where R_i is a proper subskew field of R. Then R is isomorphic to Z_2^k, the kth cartesian power of the two element field Z_2 with $k \geq 2$. Further $n = 2^k - 1$.*

References

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