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中文摘要
探討 Jacobi 場是黎曼流形上的重要課題，而且其本身在幾何與拓樸也有重要的應用。
我們證明Jacobi 場的大小在Alexaddrov空間上 不小於其相對的Jacobi 場的大小在常曲率空間上。Alexandrov 空間是黎曼流形的推廣且有重要的應用。

關鍵詞:
Jacobi 場, Alexandrov 空間, 黎曼流形, 測地線。

Jacobi fields on Alexandrov Spaces

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Abstract

Jacobi fields is a main tropics in the course of Riemannian Geometry. It also has many applications in Riemannian Geometry. Alexandrov Spaces is a natural generation of Riemannian manifolds. We prove theorems on Jacobi fields on Alexandrov Spaces and discuss how it related to geodesic and their angles. Because Alexandrov spaces do not have differentiable structure, So our methods is different from that of Riemannian manifolds. We use intrinsic means.

1 Background and Motivation

Jacobi fields is introduced by Jacobi for study whether geodesic is the shortest curve. Marston Morse latter invents Morse theory using many properties of Jacobi fields citep8. Using Morse theory many geometric and topological properties of Riemannian manifolds are reveal. Alexandrov spaces is a greater generalized of Riemannian manifolds, which inherentbasic properties from Riemannian manifolds and have many applications in geometry and combinatorics group theory and analysis [7]. Hence the study of the Jacobi fields on Alexandrov is very important and meaningful.

2 Main Results and Discussions

First we review relevant concepts related to this work. For more details concerning basic definitions and properties of spaces of curvature bounded from above, see [1, 4].

A curve in a metric space (M, ρ) is called a geodesic (segment) if its length is equal to the distance between its ends. We often use AB to denote $\rho(A, B)$ or the geodesic itself. A metric space is called geodesic if for any two points in metric space, there is a geodesic between them. We do not require geodesics to have constant speed, even though that requirement is customary in Riemannian manifolds.

By S_K we denote a surface of constant curvature K, i.e., a Euclidean plane when $K = 0$, a hyperbolic (Lobachevskii) plane when $K < 0$ and an open hemisphere of radius $1/\sqrt{K}$ when $K > 0$.

A triangle $T = \Delta ABC$ in a metric space (M, ρ) with vertices A, B, C is the union of

Let $T = \Delta ABC$ be a triangle in a metric space. A triangle $T' = \Delta A'B'C'$ in S_K is said to be a comparison triangle for $T = \Delta ABC$ if $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$. R_K domain is a metric space with the following properties

1. R_K is a geodesic space with intrinsic metric.
2. If $K > 0$, then the perimeter of each triangle in R_K is less than $2\pi/\sqrt{K}$.
3. Each triangle in R_K has nonpositive excess with respect to K, i.e.,

$$\delta(T) \leq 0. \quad (1)$$

By a space of curvature bounded from above by K in the sense of A.D. Alexandrov we understand a metric space with intrinsic metric, each point of which is contained in some neighborhood of the original space which is an R_K domain. It is well known that we can not define exponential map in Alexandrov spaces with curvature bounded from above by k. Thus follows from the paper [3, 9] we define the norm of a Jacobi field. Let $\sigma : [0, a] \to X$ be a segment and let $\alpha = \gamma_s(t)$ be a family of normal segments from p to $\sigma(s)$. It follows from Alexandrov convexity that

$$(1+\delta)|\alpha(0,t)\alpha(s,t)| \leq |\alpha(0,(1+\delta)t)\alpha(s,(1+\delta)t)|.$$

Hence for almost all t there is

$$|J| = |J|_\alpha(t) = \limsup_{h\to 0} \frac{1}{h}|\alpha(0,t)\alpha(h,t)|.$$

The main theorem we prove is the following: *by k and $\bar{M}$ is a space form with constant curvature k. Let $\gamma : [o,b] \to M$, $\bar{\gamma} : [o,b] \to \bar{M}$ be two geodesics on M and $\bar{M}$ respectively. Let $|J|$, $|\bar{J}|$ be the norm of two Jacobi fields on γ and $\bar{\gamma}$ with $|J(0)| = |\bar{J}(0)|$ and $|\dot{J}(0)| = |\dot{\bar{J}}(0)|$ respectively. Then $|J(t)| \geq |\bar{J}(t)|$.*

Proof of this theorem is purely geometric means. First we express the derivative of Jacobi norm by geometric quality i.e. angle between two Jacobi fields. Then by Alexandrov's angle comparison theorem [2] we are able to show main theorem.

Argue dually we also show that

Theorem 2.2 *Suppose that M is a Alexandrov space with curvature bounded from below by k and $\bar{M}$ is a space form with constant curvature k. Let $\gamma : [o,b] \to M$, $\bar{\gamma} : [o,b] \to \bar{M}$ be two geodesics on M and $\bar{M}$ respectively. Let $|J|$, $|\bar{J}|$ be the norm of two Jacobi fields on γ and $\bar{\gamma}$ with $|J(0)| = |\bar{J}(0)|$ and $|\dot{J}(0)| = |\dot{\bar{J}}(0)|$ respectively. Then $|J(t)| \leq |\bar{J}(t)|$.*

3 Self Evaluation

We prove main theorem using only pure geometric methods, Without using any analytic techniques. Hence the theorem is beautiful and the project is very interesting. In the future we wish to explore more topological properties of Alexandrov spaces and to see whether the topology of Alexandrov spaces are determined by curvature in the sense of Alexandrov.

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