行政院國家科學委員會補助專題研究計畫成果報告

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※ Alexandrov 空間的凸性研究 ※
※ On the convexity of Alexandrov Spaces ※
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計畫主持人：陳健雄

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□赴國外出差或研習心得報告一份

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行政院國家科學委員會專題研究計畫成果報告

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一、中文摘要

本文主要是研究以下的猜測：假設 $(M \times N, d_f)$，是一個具有非正曲率的亞力山大空間，則 M 和 N 必為兩個非正曲率空間而且 f 是凸函數。

關鍵詞：亞力山大空間、距離、凸函數、非正曲率、直積

Abstract

The main purpose of this paper is to investigate the following conjecture : suppose that warped product space $(M \times N, d_f)$ is a non-positive curved Alexandrov space, then both M and N must be non-positive curved Alexandrov space and function f is convex.

Keywords: Alexandrov space, metric, convex function, non-positive curvature, direct product

二、緣由與目的

Alexandrov 空間是由幾何大師 A. D. Alexandrov 在 1960 年代提出[2,3]，此空間是黎曼幾何的推廣，文獻又稱之為廣義黎曼幾何。在這空間上可定義測地線，兩相交曲線的角度，從而定義出曲率的觀念。此空間不像黎曼幾何需要微分結構與黎曼測距；而是回歸到幾何的本質。所以此劃時代的觀念自然應用無窮，目前此理論發展快速，特別是 M. Gromov 的努力已將此理論運用到幾何，組合群論等領域[8,9]。非正曲率黎曼空間有許多優美的定理，而存在許多凸函數在此空間上。兩者息息相關。借由凸函數的研究可了解許多非正曲率黎曼空間的性質。此計劃是考慮非正曲率亞力山大空間與凸函數之間的關係，相信經由此一研究能更深入了解亞力山大空間的幾何與拓樸性質。

三、結果與討論

First we review some definitions and notations relevant to this paper, for detail we refer to [1,4]. A curve is a continuous map from an interval to a metric space. A curve in a metric space is called a geodesic if its length is equal to the distance between its edns. A metric space is called geodesic if for any two points in metric space, there is a geodesic between them. A space with curvature bounded from above by K is by definition for each point in space there is a neighborhood such that for each triangle in this neighborhood, there is a comparison triangle in space form with constant curvature K and satisfies K-concavity property, namely let D, E be points on sides AB and Ac of the triangle T=ΔABC on space with curvature bounded from above by K and D' and E' the corresponding points on the sides of the comparison triangle $T' = \Delta A'B'C'$ on space form with constant curvature K, then $DE \le D'E'$.

The main theorem in this paper is the following:

Main Theorem. If (M, d_M) and (N, d_n) are two metric spaces with distance function d_M and d_N respectively. If warped product space $(M \times N, d'_f)$ is a non-positively curved space, then M and N are non-positively curved spaces provided function f has minimum and function f is convex. We now focus on the proof of the main theorem. We need the following lemmas:

Lemma 1. is crucial step as it related geodesic in component space to the whole space.

Lemma 1. Geodesics in M lift horizontally to geodesics in $(M \times N, d_f)$. Thus $M \times \{p\}$ is totally geodesic in $(M \times N, d_f)$ for every point $p \in N$.

Proof. The idea of proof is to show that Geodesic in M is shortest among all the curves with the fixed end points.

Lemma 2. If $p \in M$ is a local minimum point of function f, then (p, α) is a geodesic in $(M \times N, d_f)$ for every geodesic α in N.

Proof . The idea of proof is similar to that of Lemma 1.

Lemma 3. Let γ be a geodesic in M and f be a continuous function on M. Let $\alpha = \gamma \times p$ and $\beta = \gamma \times q$ be two lifts of γ. Then $\lim_{p \to q} d(\alpha(t), \beta(t)) / d_N(p, q) = f(\beta(t))$.

Proof. The proof is base on Lemma 1.

Now apply the K-convexity property, we have the following result. There is a similarily result in Riemannian geometry. But the proof is quite different. This result also show the powerful of the notation of Alexandrove space with curvature bounded from above as there is no Riemannian metric but many theorems can still be proved.

Lemma 4. The distance function between two geodesics in a non-positively curved space is convex.

Proof. It suffices to show that if AB and CD are two geodesic segments, E and F middle point of AB and CD respectively, then $EF \le \frac{1}{2}(AC + BD)$. Let $\Delta A'C'D'$ and $\Delta A'B'D'$ be the comparison triangle of ΔACD and ΔABD respectively. Let G and G' be the middle point of geodesic segment AD and $A'D'$ respectively. Then by K-convexity, we have $EG \le E'G' = \frac{1}{2} BD$ and $GF \le G'F' = \frac{1}{2} AC$. Add up these two inequality and the result follows.

Lemma 5. Suppose that (M, d) is a non-positively curved Alexandrov space , then (M, nd) is a non-positively curved Alexandrov space.

Proof of the Main Theorem. For clarity we divide the proof into three steps.

Step 1.
We shall show that M is a Alexandrov space with non-positive curvature.
We claim that every point x in M has a ball which is a Alexandrov space with non-positive curvature. Since $(M \times N, d_f)$ is a Alexandrov space with non-positive curvature, we can find a ball $B(x, r)$ in M such that $B(x, r) \times p$ is a Alexandrov space with non-positive curvature. Let T be any triangle in $B(x, r)$. Then $T \times p$ is a triangle by Lemma 1. Hence T has K-concavity

property and $B(x,r)$ is convex. It follows that M is a Alexandrov space with non-positive curvature.

Step 2.
We shall show that N is a Alexandrov space with non-positive curvature. Suppose that function f takes its minimum at point m in M Since $(M\times N,d_f)$ is a Alexandrov space with non-positive curvature, there is a ball $B(x,r)$ in $(N,d_{f(m)d_N})$ such that $m\times B(x,r)$ lies in $(M\times N,d_f)$. Let Δ be any triangle in $B(x,r)$ in $(N,d_{f(m)d_N})$. Then $m\times\Delta$ is a triangle by Lemma 2. Thus Δ has the K-concavity property and hence $B(x,r)$ in $(N,d_{f(m)d_N})$ is convex. Therefore $(N,f(m)d_N)$ is a Alexandrov space with non-positive curvature.
By Lemma 5. It follows that (N,d_N) is a Alexandrov space with non-positive curvature.
Step 3. By Lemma 4, we have $EF\le tAB+(1-t)CD$, divide this equation by $d_N(p,q)$ get

$$EF/d_N(p,q)\le tAB/d_N(p,q)+(1-t)CD/d_N(p,q).$$

Letting $p\to q$, we see that function f is convex.

四、計畫成果自評

In this paper we proved a beautiful theorem conjectured by Professor R. L. Bishop. The beauty of this theorem is that the notation of Alexandrov space, convex function are related by this Main Theorem and method used here is pure geometric. Furthermore this theorem may be applicable to general relativity [5,10]. Our method can be used in the Alexandrov spaces with lower bound, for this kind of space please refer to [6]. We achieve the goal of this project and from this point of view this project is successful.

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