行政院國家科學委員會補助專題研究計畫成果報告

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※　　Alexandrov **空間的幾何性質研究**　　　　　　※
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※　　On some properties of nonpositively　　　　※
※　　curved Alexandrov Spaces
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計畫類別：個別型計畫

計畫編號：NSC **89-2115-M-018-015**

執行期間： 89 **年** 8 **月** 1 **日至** 90 **年** 7 **月** 31 **日**

計畫主持人：陳健雄

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執行單位：彰化師大數學系

中　華　民　國　90　**年**　10　**月**　31　**日**

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一、中文摘要

本計劃的主要目的是研究下面的問題並由此問題的解答了解亞力山大空間的幾何性質：

假設 (M,d) ,(N,d’) 分別為 Alexandrov 空間，其中 M 代表實數軸 R， d與 d’ 分別為 M 與 N 空間上的距離函數。假設函數 f 為定義在實數軸的凸函數，令距離函數 e代表由 d, d’ 與 f 所誘導的距離函數。

一個很自然的問題是如果(M,d) 與(N,d’)同時為非正的 Alexandrov 空間，則由這二個空間所產生的積空間當其距離函數為 e’時是否仍為非正 Alexandrov 空間 ？

Reshetnyak **教授提出一個類似的問題。假設** $f(t) = \cos\sqrt{K}t$ **定義在一個閉區間** $I = [-\frac{®}{2\sqrt{K}}, \frac{®}{2\sqrt{K}}]$ **上，並且假設**(N,d’) **為一個有上界** K Alexandrov **空間，那麼空間** I **與空間** N **的積空間是否仍是一個有上界的** Alexandrov **空間。**

關鍵詞：亞力山大空間，距離函數，非正曲率，直積。凸函數。

Abstract

The purpose of this project is to investigate the following problems :

Suppose that (M,d) and (N,d’) are Alexandrov spaces, where M is real line R, d and d’ are distance function on M and N respenctively. Let function f be a convex function define on real line. Let distance function e be induced from d, d’ and f. It is easy to define the meaning of induced metric from
that of Riemannian manifolds. The precise definition of induced
distance function will be investigated.

The natural question one may ask is that if both (M.d) and (N,d’) are nonpositive Alexandrov spaces. Is their product space with distance function e still a nonpositive Alexandrov space ?

A similar problem raised by Professor Reshetnyak is that suppose that $f(t) = \cos\sqrt{K}t$ is define on the interval $I = [-\frac{®}{2\sqrt{K}}, \frac{®}{2\sqrt{K}}]$. Suppose that (N,d’) is Alexandrov space with curvature bounded above by K. Is their product space (I with N) still a space of curvature bounded from above?

Keywords: Alexandrov space, distance function, nonpositive curvature, product, convex function.

二、緣由與目的

非正黎曼流形較少被討論，因而相對的這方面的結果較少 Bishop and O’Neill **在文獻** 5 **証明非正黎曼流形的乘積仍為非正黎曼流形。他們所定義的乘積稱為扭曲積，其扭曲的程度由定義在第一個黎曼流形的正實函數所控制。這種乘積的優點是截面曲率可以控制而且乘積後的拓樸性質非常有趣。這種乘積在廣義相對論有很重要的應用詳細情形請參考文獻** 10**。Alexandrov 空間是由幾何大師 A. D. Alexandrov 在 1960 年代提出[2,3] ,此空間是黎曼幾何的推廣，文獻又稱之為廣義黎曼幾何。在這空**

間上可定義測地線，兩相交曲線的角度，從而定義出曲率的觀念。此空間不像黎曼幾何需要微分結構與黎曼測距；而是回歸到幾何的本質。所以此劃時代的觀念自然應用無窮，目前此理論發展快速，特別是 M. Gromov 的努力已將此理論運用到幾何，組合群論等領域[8,9]。非正曲率黎曼空間有許多優美的定理，而存在許多凸函數在此空間上。兩者息息相關。借由凸函數的研究可了解許多非正曲率黎曼空間的性質。此計劃是考慮非正曲率亞力山大空間與凸函數之間的關係，相信經由此一研究能更深入了解亞力山大空間的幾何與拓樸性質。

三、結果與討論

First we review some definitions and notations relevant to this paper, for detail we refer to [1,4]. A curve is a continuous map from an interval to a metric space. A curve in a metric space is called a geodesic if its length is equal to th e distance between its edns. A metric space is called geodesic if for any two points in metric space, there is a geodesic between them. A space with curvature bounded from above by K is by definition for each point in space there is a neighborhood such that for each triangle in this neighborhood, there is a comparison triangle in space form with constant curvature K and satisfies K-concavity property, namely let D, E be points on sides AB and Ac of the triangle T= ΔABC on space with curvature bounded from above by K and D' and E' the corresponding points on the sides of the comparison triangle $T' = \Delta A'B'C'$ on space form with constant curvature K, then $DE \le D'E'$.

The main theorem in this paper is the following:

Main Theorem. Let M be a real line and (N, d_N) has nonpositive curvature and $f: M \to R^+$ is convex, then $(M \times N, d)$ has nonpositive curvature.

This means that the Alexandrov space is preserved by the operation. In order to prove the main theorem we need the following lemmas:

Lemma 1. If $\pm = (\sim, ¢)$ is a geodesic in $(M \times N, d)$, then $¢$ is a geodesic in N.

This lemma relate the geodesic in the construted space to that of second component space. Note that the result is false for the first component space.

Lemma 2. If $¤: N \to N$ is an isometric map and $¥ = (r, s)$ is a curve in $(M \times N, d)$, Then $l(¥) = l(r, ¤(s))$.

This lemma claims that the length of constructed space is preserved by the isometric map in the second component space. Note that the result remains true for the first component space.

Lemma 3. Let (N, d_N) have nonpositive curvature and let $f: R \to R^+$ be continuous. Then for any small triangle in the constructed space, there is a comparison triangle in model space $(R \times R^2, d)$ which satifies convex property.

This Lemma is a key step to prove the Main Theorem which essential say that there are many comparison triangle in model space and make the comparison possible.

The idea of proof of the main theorem goes as follows: first we compare $(M \times N, d')$ to the model space $(M \times R^2, d)$ and this is possible by Lemma 1, Lemma 2 and Lemma 3. Next we show that $(M \times R^2, d)$ is a Alexandrov space with curvature bounded from above. Finally we verify that $(M \times N, d')$ is a Alexandrov space with curvature bounded from above. For convex function is not smooth, we use limited process along with the result of limted Alexandrov space is a Alexandrov space with proper assumption. Using the same method i n the Main Theorem, it can show that the problem proposed by Professor Reshetnyak is true.

四、計畫成果自評

In this paper we proved a beautiful theorem conjectured by Professor R. L. Bishop. The result of the main theorem can be seen as the extension of the result in riemannian manifold.The beauty of this theorem is that the notation of Alexandrov space, convex function are related by this Main Theorem and method used here is pure geometric. Furthermore this theorem may be applicable to general relativity [5,10]. Our method can be used in the Alexandrov spaces with lower bound, for this kind of space please refer to [6]. We achieve the goal of this project and from this point of view this project is successful.

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