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A New Proof of a Box-stacking Theorem*

Kuo-Jye Chen†

Department of Mathematics,
National Changhua University of Education,
Changhua 50007, Taiwan

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Abstract

We give a direct proof of a recent generating function identity of Andrews and Sellers on box stacking. Our method provides alternate proofs for other related identities.

Keywords and Phrases: *Partitions, Partition analysis, Non-squashing partitions, Stacking boxes, Abel's identity.*

Recently Andrews and Sellers [1] considered the following box-stacking problem of Sloane:

> We are given n boxes, labeled $1, 2, \cdots, n$. For $i = 1, \cdots, n$, box i weigh $(m-1)i$ grams (where $m \geq 2$ is a fixed integer) and box i can support a total weight of i grams. What is the number $a_m(n)$ of different ways to build a single stack of boxes in which no box will be squashed by the weight of the boxes above it?

They derived the following nice generating function of $a_m(n)$ via MacMahon's partition analysis:

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†E-mail: kjchen@cc.ncue.edu.tw

Theorem 1.([1, Theorem 1.1]) *For* $m \geq 3$,

$$\sum_{n=0}^{\infty} a_m(n) q^n = \frac{1}{(1-q)^2 \prod_{i=0}^{\infty}(1-q^{(m-1)m^i})}.$$

The purpose of this note is to give a direct proof Threorem 1. Before we go about proving it, we prove two lemmas regarding m-non-squashing partitions. An m-non-squashing partition is a partition $p_k + p_{k-1} + \cdots + p_1$ with $p_k \geq p_{k-1} \geq \cdots \geq p_1 \geq 1$ and $p_i \geq (m-1)(p_{i-1} + p_{i-2} + \cdots + p_1)$, for $i = k, k-1, \cdots 2$. The first lemma appears in [3]. For the sake of completeness, we give a different presentation.

Lemma 1. *For* $k = 0, 1, 2, 3, \cdots$, *define*

$$f_k(q) = \sum_{\substack{p_k \geq p_{k-1} \geq \cdots \geq p_1 \geq 1 \\ p_i \geq (m-1)(p_{i-1} + p_{i-2} + \cdots + p_1),\ i = k, k-1, \cdots, 2}} q^{p_k + p_{k-1} + \cdots + p_1}.$$

Then $f_0(q) = 1$, $f_1(q) = \frac{q}{1-q}$, *and, for* $k \geq 2$,

$$f_k(q) = \frac{q^{m^{k-1}}}{(1-q)(1-q^m)(1-q^{m^2})\cdots(1-q^{m^{k-1}})}.$$

Proof. By definition, we have $f_0(q) = 1$ and

$$f_1(q) = \sum_{p_1 \geq 1} q^{p_1} = \frac{q}{1-q}.$$

For $k \geq 2$, we have

$$\begin{aligned} f_k(q) &= \sum_{\substack{p_k \geq 0 \\ p_{k-1} \geq p_{k-2} \geq \cdots \geq p_1 \geq 1 \\ p_i \geq (m-1)(p_{i-1} + p_{i-2} + \cdots + p_1),\ i = k-1, k-2, \cdots, 2}} q^{p_k + m(p_{k-1} + p_{k-2} + \cdots + p_1)}. \\ &= \frac{1}{1-q} \cdot f_{k-1}(q^m). \end{aligned}$$

Iteration of this recurrence gives

$$\begin{aligned} f_k(q) &= \frac{1}{1-q}\cdot f_{k-1}(q^m) \\ &= \frac{1}{1-q}\cdot\frac{1}{1-q^m}\cdot f_{k-2}(q^{m^2}) \\ &= \cdots \\ &= \frac{1}{(1-q)(1-q^m)\cdots(1-q^{m^{k-2}})}\cdot f_1(q^{m^{k-1}}) \\ &= \frac{q^{m^{k-1}}}{(1-q)(1-q^m)(1-q^{m^2})\cdots(1-q^{m^{k-1}})}, \end{aligned}$$

which completes the proof of Lemma 1. □

Lemma 2. *For $k = 0, 1, 2, 3, \cdots$, define*

$$g_k(q) = \sum_{\substack{p_k \geq p_{k-1} \geq \cdots \geq p_1 \geq 1 \\ p_i \geq (m-1)(p_{i-1}+p_{i-2}+\cdots+p_1),\ i=k,k-1,\cdots,2}} q^{p_k}.$$

Then $g_0(q) = 1$, $g_1(q) = \frac{q}{1-q}$, $g_2(q) = \frac{q^{m-1}}{(1-q)(1-q^{m-1})}$, and, for $k \geq 3$,

$$g_k(q) = \frac{q^{(m-1)m^{k-2}}}{(1-q)(1-q^{m-1})(1-q^{(m-1)m})(1-q^{(m-1)m^2})\cdots(1-q^{(m-1)m^{k-2}})}.$$

Proof. By definition, we have $g_0(q) = 1$,

$$g_1(q) = \sum_{p_1 \geq 1} q^{p_1} = \frac{q}{1-q},$$

$$g_2(q) = \sum_{p_2 \geq p_1 \geq 1,\ p_2 \geq (m-1)p_1} q^{p_2} = \sum_{p_2 \geq 0,\ p_1 \geq 1} q^{p_2+(m-1)p_1} = \frac{q^{m-1}}{(1-q)(1-q^{m-1})},$$

and, for $k \geq 3$, we have

$$\begin{aligned} g_k(q) &= \sum_{\substack{p_k \geq 0 \\ p_{k-1} \geq p_{k-2} \geq \cdots \geq p_1 \geq 1 \\ p_i \geq (m-1)(p_{i-1}+p_{i-2}+\cdots+p_1),\ i=k-1,k-2,\cdots,2}} q^{p_k+(m-1)(p_{k-1}+p_{k-2}+\cdots+p_1)} \\ &= \frac{1}{1-q}\cdot f_{k-1}(q^{m-1}) \\ &= \frac{1}{1-q}\cdot\frac{q^{(m-1)m^{k-2}}}{(1-q^{(m-1)})(1-q^{(m-1)m})(1-q^{(m-1)m^2})\cdots(1-q^{(m-1)m^{k-2}})}, \end{aligned}$$

where in the last equality we have used Lemma 1. The lemma follows. □

Now we are in a position to prove Theorem 1.

Proof of Theorem 1. It is observed in [1, Section 2] that $a_m(n)$ is equal to the total number of m-non-squashing partitions with parts $\le n$. Thus

$$\begin{aligned}
\sum_{n=0}^{\infty} a_m(n)q^n &= \sum_{n=0}^{\infty} q^n \sum_{\substack{n \ge p_k \ge p_{k-1} \cdots \ge p_1 \ge 1,\ k \ge 0 \\ p_i \ge (m-1)(p_{i-1}+p_{i-2}+\cdots+p_1),\ i=k,k-1,\cdots,2}} 1 \\
&= \sum_{\substack{p_k \ge p_{k-1} \cdots \ge p_1 \ge 1,\ k \ge 0 \\ p_i \ge (m-1)(p_{i-1}+p_{i-2}+\cdots+p_1),\ i=k,k-1,\cdots,2}} \sum_{n \ge p_k} q^n \\
&= \frac{1}{1-q} \cdot \sum_{\substack{p_k \ge p_{k-1} \cdots \ge p_1 \ge 1,\ k \ge 0 \\ p_i \ge (m-1)(p_{i-1}+p_{i-2}+\cdots+p_1),\ i=k,k-1,\cdots,2}} q^{p_k} \\
&= \frac{1}{1-q} \cdot \{g_0(q)+g_1(q)+g_2(q)+g_3(q)+g_4(q)+\cdots\} \\
&= \frac{1}{1-q} \cdot \{1 + \frac{q}{1-q} + \frac{q^{m-1}}{(1-q)(1-q^{m-1})} + \frac{q^{(m-1)m}}{(1-q)(1-q^{m-1})(1-q^{(m-1)m})} + \\
&\qquad + \frac{q^{(m-1)m^2}}{(1-q)(1-q^{m-1})(1-q^{(m-1)m})(1-q^{(m-1)m^2})} + \cdots\} \\
&= \frac{1}{1-q} \cdot \frac{1}{(1-q)(1-q^{m-1})(1-q^{(m-1)m})(1-q^{(m-1)m^2})\cdots},
\end{aligned}$$

where the penultimate equality follows from Lemma 2 and the last equality follows by adding from the first term in the parenthesis. The result of the theorem follows. □

I note that an alternate proof of the main result of Andrews and Sellers [1] has been given by Rødseth [4].

We close by remarking that via Abel's identity [2, Eq.(1)] our method provides similar proofs for the main theorems in [3] and [6] and for the identity in [5, Eq. (4)]. We leave the details to the interested reader.

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