PROSPECT THEORY AND DECISION WEIGHTS

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ABSTRACT

Proposed by Kahneman and Tversky as an alternative model for analyzing choice under risk and uncertainty, prospect theory is characterized by a value function and a probability-weighting function which overweights small probabilities but underweights high and moderate probabilities. Since the probability-weighting function plays a crucial role in prospect theory and its applicability in various areas, the paper analyzes the correspondence between the probability-weighting function and the probability function embedded in choice under risk and uncertainty, and the properties of low-probabilities overweighting and subcertainty associated with the correspondence.

Keywords: risk; uncertainty; subcertainty; overweighting; prospect theory

1. INTRODUCTION

Daniel Kahneman and Amos Tversky, two Israeli psychologists already famous in the field of judgment heuristics, as mentioned by Barberis (2013), published in the journal *Econometrica* in 1979 a paper entitled "Prospect theory: an analysis of decision under risk." With regard to their influential work, Barberis stated:

> More than 30 years later, prospect theory is still widely viewed as the best available description of how people evaluate risk under experimental settings. Kahneman and Amos Tversky's papers have been cited tens of thousands of times and were decisive in awarding Kahneman the Nobel Prize in economic sciences in 2002. (Tversky would surely have shared the prize had he not passed away in 1996 at the age of 59) (p. 173).

This inspiring paper "describes several classes of choice problems in which preferences systematically violate the axioms of expected utility theory" (Kahneman and Tversky, 1979, p. 263). Kahneman and Tversky argued that "utility theory, as it is commonly interpreted and applied, is not an adequate model and we propose an alternative account of choice under risk" (p. 263). As mentioned by Kahneman and Tversky (1992) with regard to the theory, they "presented a model of choice called prospect theory, which explained the major violations of expected utility theory in choices between risky prospects with a small number of outcomes (Kahncman and Tversky, 1979; Tversky and Kahneman, 1986) (p. 297). They stated that this theory is characterized by a value function and a nonlinear transformation of the probability scale in that a value function is steeper for losses than for gains, and concave for gains, but convex for losses, while a nonlinear transformation of the probability scale overweights small probabilities, but underweights high and moderate probabilities.

As mentioned by Tversky and Kahneman (1992) with regard to later research reports pertinent to prospect theory, “in an important later development, several authors (Ouiggin, 1982; Schmeidler, 1989; Yaari, 1987; Weymark, 1981) have advanced a new representation, called the rank-dependent or the cumulative functional, that transforms cumulative rather than individual probabilities” (p. 298). In response to the academic development and to extend the original version of prospect theory, which was established with the aid of evidence regarding choices between risky prospects containing merely a small number of outcomes, Kahneman and Tversky presented in their article “a new version of prospect theory that incorporates the cumulative functional and extends the theory to uncertain as well to risky prospects with any number of outcomes” (p. 298) and called the resulting model cumulative prospect theory.

While the influence of prospect theory is far-reaching, not only academically but also practically, its potential impact on later academic development can be described by the following statements presented by Barberis (2013):

> It is curious, then, that so many years after the publication of the 1979 paper, there are relatively few well-known and broadly accepted applications of prospect theory in economics. One might be tempted to conclude that, even if prospect theory is an excellent description of behavior in experimental settings, it is less relevant outside the laboratory. In my view, this lesson would be incorrect. Rather, the main reason that it has taken so long to apply prospect theory in economics is that, in a sense that I make precise in the next section, it is hard to know *how* to apply it. While prospect theory contains many remarkable insights, it is not ready-made for economic applications (pp. 173-174).

Since the probability-weighting function plays a crucial role in prospect theory and its applicability in various areas, the present paper is to analyze the correspondence between the probability-weighting function and the probability function embedded in prospect theory, and the properties of low-probabilities overweighting and subcertainty associated with the correspondence.

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2. THE PROBABILITY-WEIGHTING FUNCTION

As proposed by Kahneman and Tversky (1979), a prospect $(x_1, p_1;...;x_n, p_n)$ can be described as a contract which yields outcome x_i with probability p_i, where $\sum_i p_i = 1$ for all outcomes, and decision making under risk concerns a choice between prospects. In prospect theory, an outcome is associated with two important elements, a value yielded by the outcome and a decision weight assigned by a decision maker. Inferred from choices between prospects, decision weights, they argued, measure not merely the likelihood of events perceived by a decision maker, but also the impact events may have on the desirability of prospects. A decision maker assigns decision weights to stated probabilities through a weighting function, whereas "the decision weights are not probabilities" in that "they do not obey the probability axioms and they should not be interpreted as measures of degree or belief" (Kahneman and Tversky ,1979, p. 280). The weighting function π is presented by Kahneman and Tversky as an increasing function of p, with the salient properties $\pi(0) = 0$, and

$\pi(1)=1$. The outcomes of a strictly positive prospect are all positive, while those of a strictly negative prospect are all negative . If a prospect is neither strictly positive nor strictly negative, it is referred to as a regular prospect. In the terms of Kahneman and Tversky, the over-all value of a regular prospect can be represented as $V(x_1, p_1;...;x_n, p_n)=\sum_{i=1}^{n}\pi(p_i)\upsilon(x_i)$, with $\upsilon(0)=0$ for the value function and $\pi(0)=0$ and $\pi(1)=1$ for the weighting function π.

While Kahneman and Tversky (1979) asserted that the decision weights assigned by a decision maker measure not merely the likelihood of events perceived by the decision maker, but also the impact events may have on the desirability of prospects, they proposed strict relations between the weighting function π and the probability function p by assuming that the weighting function π is an increasing function of p and is absolutely continuous with respect to p, which is indicated henceforth by $\pi << p$. The impact events may have on the desirability of prospects appeared not to have been addressed in the specification of the weighting function so thoroughly as Kahneman and Tversky asserted, which necessitates further theoretical and empirical investigation in this respect. If the weighting function π and the subjective probability function p possessed by a decision maker have the properties of a finite measure, by Radon-Nikodym theorem, for the two σ-finite measures on $(\Omega, \Im)$ such that $\pi << p$, there would exit a nonnegative density function f such that $\pi(E)=\int_E fdp$ for all $E\in\Im$, and for two such densities f and g, $p[f\neq g]=0$. The assumption of $\pi << p$ indicates that the weighting function π is dominated by the probability function p, and the existence of such a unique nonnegative density function f suggests the tight link of the weighting function π to the probability function.

From the perspective of a correlating device $\{\Omega, \{E_i\}, P\}$ as described in Fudenburg and Tirole (1991), the structure of the information possessed by a decision maker is represented by the information partition $\{E_i\}$ as a partition of the finite state space Ω, where E_i consists of those states regarded by a decision maker as possible with the true state. In the case that a decision maker weights equally the probability of each $\omega\in E_i$, the unique nonnegative density function f under the assumption of $\pi << p$ may be further represented by a simple real function with finite range as $f=\sum_i \alpha_i I_{E_i}$, where the constant α_i may be regarded as the weighting rate for the event E_i and I_{E_i} is the indicator function of E_i, assuming the value 1 on E_i and 0 on E_i^c. In this framework, the weighting function π can be represented as $\pi(E_i)=\int_{E_i}\sum_i \alpha_i I_{E_i} dp$, and $\pi(E_i)=\alpha_i p(E_i)$. The relation between the weighting function π and the probability function p is characterized by the weighting rates $\{\alpha_i\}$ corresponding to the information partition $\{E_i\}$ of the correlating device $\{\Omega, \{E_i\}, P\}$. To avoid the dominance of the probability function p over the weighting function π and to take into consideration the impact events may have on the desirability of prospects, the assumption of $\pi << p$ may be relaxed by including a desirability-representing measure $\lambda(E_i)$ in the specification of the weighting function π in such a form as $\pi(E_i)=\lambda(E_i)+\int_{E_i}\sum_i \alpha_i I_{E_i} dp$. The properties and appropriate form of a weighting function π as such, with a desirability-representing measure included in it, needs to be thoroughly considered for the over-all value function to possess desirable properties for individual choice under risk and uncertainty.

3. LOW-PROBABILITIES OVERWEIGHTING

With regard to the properties of the weighting function, Kahneman and Tversky (1979) proposed that for very low probabilities p, they are generally overweighed by the weighting function so that $\pi(p) > p$. The proposition is derived from the evidence revealed by the responses of university faculty and students to some hypothetical choice problems presented to them. As presented by Kahneman and Tversky, for symmetric fair bets $(y,.5;-y,.5)$ and $(x,.5;-x,.5)$ with $x > y \geq 0$, the averseness would generally increase with the size of the outcome, and thus symbolically $\upsilon(y)+\upsilon(-y) > \upsilon(x)+\upsilon(-x)$, which yields $\upsilon(x) < -\upsilon(-x)$ by setting $y=0$ and $\upsilon'(x) < \upsilon'(-x)$ by letting y approach x. This indicates that the value function for losses is steeper than that for gains. With a larger proportion of respondents choosing (5,000, .001) instead of (5), by further assuming a concave graph for the value function for gains, Kahneman and Tversky showed that $\pi(.001) > \upsilon(5)/\upsilon(5{,}000) > .001$.

The empirical evidence presented by Kahneman and Tversky (1979) regarding choices between risky prospects reveals the overweighting by a weighting function of low probabilities associated by a probability function with those events which are less likely to take place. Theoretically, for two totally finite measures μ and ν such that $\nu << \mu$ and ν is not identically zero, "there exists a positive number ε and a measurable set A such that $\mu(A) > 0$ and such that A is a positive set for the signed measure $\nu - \varepsilon\mu$." (Halmos, 1950, p. 128). If the weighting function π and the probability function p possessed by a decision maker have the properties of a finite measure, for the two totally σ-finite measures such that $\pi << p$ and π is not identically zero, there would exist a positive number ε and a measurable set $\overline{E}$ such that $p(\overline{E}) > 0$ and such that $\pi(E \cap \overline{E}) - \varepsilon p(E \cap \overline{E}) \geq 0$ for every measurable set E. The existence of such a positive number ε and a measurable set $\overline{E}$ indicates an infimum for the weighting of probabilities associated with events contained in the measurable set $\overline{E}$, which can be expressed by $\pi(E \cap \overline{E}) / p(E \cap \overline{E}) \geq \varepsilon$. If the weighting function π and the subjective probability function p possessed by a decision-maker have the properties of a finite measure, the assertion presented by Kahneman and Tversky regarding the overweighting of low probabilities by a weighting function π suggests that there would exist a positive number $\varepsilon > 1$ and a measurable set $\overline{E}$ such that $p(\overline{E}) > 0$ and $p(\overline{E}_k)$ is low for every $E_k \subset \overline{E}$ and such that $\pi(E \cap \overline{E}) / p(E \cap \overline{E}) > 1$ for every measurable set E.

As mentioned previously, in the framework with a correlating device $\{\Omega, \{E_i\}, P\}$ and a decision maker weighting equally the probability of each $\omega \in E_i(\omega)$, if the weighting function π and the subjective probability function p possessed by a decision maker have the properties of a finite measure, the weighting function π can be represented as $\pi(E_i) = \int_{E_i} \sum_i \alpha_i I_{E_i} dp$, and $\pi(E_i) = \alpha_i p(E_i)$, where the constant α_i may be viewed as the weighting rate which the weighting function associates with E_i. An overweighting pattern exists if $\alpha_i > 1$, while an underweighting pattern exists if $\alpha_i < 1$. As $\alpha_i = 1$, the weighting function may be viewed as unbiased.

4. SUBCERTAINTY

As Kahneman and Tversky (1979) argued, evidence indicates that the weighting function is characterized by the property of subcertainty, that is, for all $0 < p < 1$, $\pi(p) + \pi(1-p) < 1$, whereas the weighting function possesses the property of low-probabilities overweighting. The observed phenomenon of subcertainty provides information with regard to a decision maker's summation of the individual decision weights associated with all the distinct events comprising the whole state space and reveals another quite distinct characteristic possessed by the weighting function in contrast to Kahneman and Tversky's assertion that $\pi(1) = 1$. As implied by the property of subcertainty and the property that $\pi(1) = 1$, it can be established that $\pi(p(E_i)) + \pi(p(E_i^c)) \neq \pi(\Omega)$ or $\sum_{i=1}^{n} \pi(p_i(E_i)) \neq \pi(\Omega)$, where for the disjoint sequence of $(E_1, ..., E_n)$, $\cup_{i=1}^{n} E_i = \Omega$. Thus, the evidence presented by Kahneman and Tversky indicated that the weighting function possessed by a decision maker may not possess the property of countable additivity and would not be a finite measure in this respect.

In the study of Tversky and Kahneman (1992), they attempted to structurally refine the weighting function "in terms of the concept *of capacity* (Choquet, 1955), a nonadditive set function that generalizes the standard notion of probability. A capacity W is a function that assigns to each $A \subset S$ a number $W(A)$ satisfying $W(\phi) = 0$, $W(S) = 1$, and $W(A) \geq W(B)$ whenever $A \supset B$" (p. 300). However, even with further refinements of the model, the violation of the property of countable additivity by the weighting function and other critical elements characterizing the theory may make prospect theory "not ready-made for economic applications" (Barberis, 2013, p. 174) in that the theory may not be readily linked to concepts and analytical frameworks available in relevant areas and may need further elaboration for integrated applications. Questions may be raised: Is the phenomenon of subcertainty common or serious in a large sample? For those individuals who attempt to act rationally in choice under risk and uncertainty even though their actions may not be rational in all respects, would theories built on the assumption of rationality better describe their behaviors in choice under risk and uncertainty than other theories?

While such properties of subcertainty and low-probabilities overweighting possessed by the weighting function as noted by Kahneman and Tversky (1979) would provide an evidence-based perspective from which the probabilities-weighting behavior of an individual may be understood, the properties appear to be derived to a great extent from evidence on individuals' choices among prospects containing two or a small number of outcomes and may not be valid or compatible in individuals' choices among prospects containing a larger number of outcomes under specific contexts. For instance, suppose that the information partition $(E_1, E_2, ..., E_{n-1}, E_n)$ possessed by a decision maker is a finite, countable partition of the state space Ω, and for $i = 1, ..., n-1$, the value of $p(E_i)$ is low and positive. With the information partition, those events with measure 0 are ignored in the decision maker's formation of relevant choices. Under the proposition that low probabilities are overweighed, $\pi(p(E_i)) > p(E_i)$ for $i = 1,...,n-1$, and thus $\sum_{i-1}^{n-1}\pi(p(E_i)) > \sum_{i=1}^{n-1} p(E_i)$. For an information partition $(E_1, E_2,..., E_{n-1}, E_n)$ containing a large number of distinct events with E_i, $i = 1,...,n-1$, associated with low probabilities and an event E_n with a large probability $p(E_n)$ and a corresponding decision weight $\pi(p(E_n))$, the value of which is close to $p(E_n)$, it would be very likely to be true that $\pi(p(E_n)) + \sum_{i-1}^{n-1}\pi(p(E_i)) > p(E_n) + \sum_{i=1}^{n-1} p(E_i) = 1$, as the number of the distinct events with low probabilities becomes large, which implies the violation of subcertainty. On the contrary, suppose that the property of subcertainty is true and $\pi(p(E_n)) + \sum_{i-1}^{n-1}\pi(p(E_i)) < p(E_n) + \sum_{i=1}^{n-1} p(E_i) = 1$. Thus, in the presence of an event E_n with a large probability $p(E_n)$ and a corresponding decision weight $\pi(p(E_n))$ with a value close to $p(E_n)$, it would be very likely to be true with the property of subcertainty that for some subset(s) E_i, $i \in \{1,...,n-1\}$, $\pi(p(E_i)) < p(E_i)$, as the number of the low-probability events becomes large. This suggests the likelihood of the existence of incompatibility between subcertainty and low-probabilities overweighting in choice under risk and uncertainty.

5. CONCLUDING REMARKS

Prospect theory has provided an inspiring perspective from which choice under risk and uncertainty may be understood. As an enormous number of studies have undertaken with great interest to address issues pertinent to this influential theory, there still exist unexplored issues or unanswered puzzles related to the evidence-based theory, which was proposed with an attempt to pave the way for better understanding and representing human psychological operations in choice under risk and uncertainty. This paper has addressed some critical issues regarding the correspondence between the weighting function and the probability function presented in prospect theory for the purpose of enhancing teaching and learning of this influential theory and drawing more research attention to the exploration of human decision-making under risk and uncertainty.

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