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THE STABILITY AND GROWTH FOR THE PERTURBED NONLINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS

BY

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Abstract. The nonlinear variation of constant formula, Banach contraction principle and some generalizations of Gronwall's inequality are used to study the stability and growth of perturbed nonlinear systems of differential equations allowing more general perturbations than were previously allowed.

1. **Introduction.** In this paper the relations between the solutions of unperturbed system

$$x' = f(t, x) \tag{1}$$

and the solutions of the perturbed system

$$y' = f(t, y) + g(t, y, Ty) \tag{2}$$

are considered. Here x, y, f and g are elements of R^n, an n-dimensional Euclidean space, and the prime will always denote differentiation with respect to t. Let I be the interval $0 \leqq t < \infty$, R_+ be the set of positive real numbers, D be a region in R^n and $C[X, Y]$ denote the space of continuous functions from X to Y where X and Y are any convenient spaces. We shall assume that $f \in C[I \times R^n, R^n]$, that $f_x(t, x)$ exists and is continuous on $I \times R^n$, that $g \in C[I \times R^n \times R^n, R^n]$, and that T is a continuous operator which maps R^n into R^n.

The following two problems are discussed and solved in this paper:

Problem 1. The stability of the solutions of system (2).

Problem 2. The growth of the solutions of system (2).

Each of the above problems is important from both the

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theoretical and the pratical viewpoints. Recently many authors have investigated these problems. In this paper we wish to study perturbations which are stable, we also discuss perturbations of the classes of unstable systems, namely, those whose solutions grow more slowly than any positive exponential. The nonlinear variation of constant formula [1], Banach contraction mapping principal and the integral inequalities recently established by Pachpatte [13] are used in solving these problems. In this paper we shall discuss these problems under suitable conditions on g, T, f, the solution of system (1) and the matrix $\phi(t, t_0, y_0)$.

2. **Preliminaries.** Let $t_0 \geqq 0$, and let $x(t, t_0, x_0)$ denote the solution of (1) through the point (t_0, x_0) and $y(t, t_0, y_0)$ denote the solution of (2) through (t_0, y_0). Let us assume $f(t, 0) = 0$ for $t \geqq 0$ so that $x(t, t_0, 0) = 0$. It is known [7] that the derivative matrix

$$\phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

exists, and satisfies the variational system

$$Z' = f_x(t, x(t, t_0, x_0))Z,$$

$\phi(t_0, t_0, x_0) = E$ (identity matrix) and

$$\frac{\partial x(t, t_0, x_0)}{\partial t_0} = -\phi(t, t_0, x_0) f(t_0, x_0)$$

The symbol $|\cdot|$ will denote some convenient norm on R^n as well as a corresponding consistent matrix norm. We let $BC[0, \infty)$ denote the collection of all bounded continuous functions from $[0, \infty)$ into R^n with the "sup" norm defined by $\|h\| = \sup_{t\geq 0}|h(t)|$ for $h \in BC[0, \infty)$. In this paper we need the following definitions and lemmas.

DEFINITION 2.1 [5]. The solution $x = 0$ of (1) is said to be globally uniformly stable in variation if there exists a positive constant M such that

(3) $$|\phi(t, t_0, x_0)| \leqq M, \quad \text{for all } t \geqq t_0 \geq 0 \text{ and } |x_0| < \infty.$$

DEFINITION 2.2 [4]. The solution $x = 0$ of (1) is said to be exponentially asymptotically stable, if there exist constants $M > 0$ and $C > 0$ such that

$$(4) \qquad |x(t, t_0, x_0)| \leqq M|x_0| \exp(-c(t-t_0)), \ (t \geqq t_0),$$

provided that $|x_0|$ is sufficiently small.

DEFINITION 2.3 [5]. The matrix $\phi(t, t_0, x_0)$ is said to be uniformly slowly growing if, and only if, for every $\varepsilon > 0$ there exists a constant K, possibly depending on ε, such that

$$(5) \qquad |\phi(t, t_0, x_0)| \leqq K \exp(\varepsilon(t-t_0)), \ t \geqq t_0 \geqq 0, \ |x_0| < \infty .$$

DEFINITION 2.4 [5]. The type number of a vector-valued function $z(t)$ is

$$(6) \qquad \tau = \limsup_{t\to\infty} \frac{\log |z(t)|}{t} .$$

If $\tau \leq 0$, the function $z(t)$ is said to be slowly growing.

It is easy to see that a function $z(t)$ is slowing growing if, and only if, for every $\varepsilon > 0$ there exists a constant K, which may depend on ε, such that

$$(7) \qquad |z(t)| \leqq K \exp(\varepsilon\, t), \qquad t \geq 0 .$$

DEFINITION 2.5 [15]. System (1) will be called stable, if for any two solution $x(t, t_0, x_0)$ and $\bar{x}(t, t_0, \bar{x}_0)$ of (1) such that $|x_0 - \bar{x}_0| < \delta$ implies $|x(t, t_0, x_0) - \bar{x}(t, t_0, \bar{x}_0)| < C\delta$, ($C$ = constant) for $\delta > 0$ and $t \geqq t_0 \geq 0$. If further $\lim_{t\to\infty}|x(t, t_0, x_0) - \bar{x}(t, t_0, \bar{x}_0)| = 0$, the system (1) is said to be asymptotically stable.

DEFINITION 2.6 [18]. We call the solution $x = 0$ of (1)

(i) stable if for every $\varepsilon > 0$, and every $t_0 \geqq 0$, there exists $\delta(\varepsilon, t_0) > 0$ such that $|x_0| < \delta$ and $t \geq t_0$ imply $|x(t, t_0, x_0)| < \varepsilon$.

(ii) uniformly stable if (1) holds with δ independent of t_0.

(iii) asymptotically stable if for each $t_0 \geq 0$ there is a $\delta(t_0) > 0$ such that $|x_0| < \delta$ implies $|x(t, t_0, x_0)| \to 0$ when $t \to \infty$.

LEMMA 1 [3]. *If $y_0 \in D$, then for all $t \geqq t_0$, such that $x(t, t_0, y_0) \in D$, $y(t, t_0, y_0) \in D$, we have*

$$(8) \qquad \begin{aligned} &y(t, t_0, y_0) - x(t, t_0, y_0) \\ &\quad = \int_{t_0}^{t} \phi(t, s, y(s, t_0, y_0)) g(s, y(s, t_0, y_0))\, ds \end{aligned}$$

LEMMA 2 [3]. *If x_0, y_0 are in a convex subset $\hat{D}$ of D, then for* *all t for which every solution of* (1) *with initial value in* $\hat{D}$ *at* t_0 *remain in* D,

$$(9)\qquad |x(t, t_0, y_0) - x(t, t_0, x_0)| \leqq |y_0 - x_0| \sup_{m\in D} |\phi(t, t_0, m)| .$$

LEMMA 3 [4]. *If* x_0, y_0 *are in a convex subset* $\hat{D}$ *of* D, *then for all t such that every solution of* (1) *with initial values in* $\hat{D}$ *at* t_0 *remains in* D *and such that* $y(t, t_0, y_0) \in D$,

$$|y(t, t_0, y_0) - x(t, t_0, x_0)| \leqq |y_0 - x_0| \sup_{m\in D} |\phi(t, t_0, m)| + \int_{t_0}^{t} |\phi(t, s, y(s, t_0, y_0))| \, |g(s, y(s, t_0, y_0))| \, ds.$$

LEMMA 4 [13]. *Let* $u(t)$, $v(t)$, *and* $q(t)$ *be real valued nonnegative continuous functions defined on* I, *for which the inequality*

$$(10)\qquad \begin{aligned} u(t) &\leqq u_0 + \int_0^t v(t)\, u(t)\, dt \\ &\quad + \int_0^t v(s)\Big(\int_0^s q(m) u(m)\, dm\Big)\, ds , \qquad t \in I \end{aligned}$$

holds, where u_0 *is nonnegative constant, then*

$$(11)\qquad u(t) \leqq u_0 \Big(1 + \int_0^t v(s) \exp\Big(\int_0^s (v(m) + q(m))\, dm\Big)\, ds, \qquad t \in I.$$

LEMMA 5 [4]. *Suppose that the solution* $x = 0$ *of* (1) *is exponentially asymptotically stable, and that there exists* $A > 0$ *such that*

$$|g(t, y)| \leqq \lambda(t) \qquad (t \geqq 0,\ |y| < A)$$

where $g \in C[I \times R^n, R^n]$, $\lambda \in C[I, R_+]$ *and suppose that* $h \in C[I \times R^n, R^n]$, $h(t, y) = o(|y|)$ *as* $|y| \to 0$ *uniformly in t. Then there exists* $T > 0$ *such that every solution* $y(t)$ *of*

$$y' = f(t, y) + g(t, y) + h(t, y)$$

for which $|y(t_0)|$ *is sufficiently small for any* $t_0 \geqq T$ *tends to zero as* $t \to \infty$.

Following the similar argument as in the proof of Theorem 1 [13], we have

LEMMA 6. *Let* $u(t)$, $h(t)$ *and* $q(t)$ *be real-valued nonnegative continuous functions defined on* I, *for which the inequality*

$$u(t) \leqq u_0 + \int_{t_0}^{t} h(s)\, u(s)\, ds$$
$$+ \int_{t_0}^{t} h(s)\Big(\int_{t_0}^{s} q(m)\, u^p(m)\, dm\Big)\, ds, \quad t \in I, \quad t \geqq t_0 \geq 0$$

holds, where u_0 *is a nonnegative constant and* $p \geqq 0$, $p \neq 1$ *and*

$$u_0^{1-p} + (1-p)\int_{t_0}^{t} q(s)\exp\Big((p-1)\int_{t_0}^{s} h(m)\, dm\Big)\, ds > 0,$$
$$t \in I, \quad t \geq t_0 \geq 0$$

then

$$u(t) \leqq u_0 + \int_{t_0}^{t} h(s)\exp\Big(\int_{t_0}^{s} h(m)\, dm\Big)$$
$$\cdot \Big[u_0^{1-p} + (1-p)\int_{t_0}^{s} q(m)\exp\Big((p-1)\int_{t_0}^{m} h(v)\, dv\Big) dm\Big]^{1/1-p} ds$$

for all $t \in I$, *and* $t \geqq t_0 \geq 0$.

REMARK. For the case $0 \leqq p < 1$ is due to Pachpatte [13].

3. **Main results.** Following the similar argument as in the proof of Theorem 1 [14], We have

THEOREM 1. *Let the following hypothesis hold:*

(1) *Suppose that*

$$g(t, y, Ty) = h(t, y),$$

where $h \in C[I \times R^n, R^n]$.

(2) $h(t, 0) = 0$

(3) $|\phi(t, s, y(s))\, h(s, y(s)) - \phi(t, s, \bar{y}(s))h(s, \bar{y}(s))| \leqq \lambda(s)|y(s) - \bar{y}(s)|$ *and*

$$\int_0^\infty \lambda(t)\, dt < u$$

for some $u \in (0, 1)$, *where* $\lambda \in C[I, R_+]$. *Then for every* $\varepsilon > 0$, *and for any solution* $x(t)$ *of* (1) *such that* $\|x\| < (1-u)\varepsilon$, *there exists a unique solution* $y(t)$ *of* (2) *satisfying* $\|y\| < \varepsilon$.

Proof. We define

$$S(\varepsilon) = \{y : y \in BC[0, \infty),\ \|y\| \leqq \varepsilon\},$$

and we will further assume that if $y \in S(\varepsilon)$, then $y(t) \in D$, for all $t \geqq 0$. Let us define the operator F by the relation

$$Fy(t) = x(t) + \int_{t_0}^{t} \phi(t, s, y(s))\, h(s, y(s))\, ds, \qquad t \geqq t_0 \geqq 0,$$

for $y \in S(\varepsilon)$, whose fixed point corresponds to the solution of the system (2). Then $|Fy(t)| \leqq |x(t)| + \int_{t_0}^{t} \lambda(s)|y(s)|ds = (1-u)\varepsilon + \varepsilon \cdot \int_{t_0}^{t} \lambda(s)\, ds < (1-u)\varepsilon + u\varepsilon = \varepsilon$. Hence F maps $S(\varepsilon)$ into itself, on the other hand, if $y, \bar{y} \in S(\varepsilon)$, we have

$$|Fy(t) - F\bar{y}(t)| \leqq \int_{t_0}^{t} \lambda(t)\|y - \bar{y}\|\, dt$$

Therefore

$$\|Fy - F\bar{y}\| \leqq \left(\int_{t_0}^{t} \lambda(t)\, dt\right)\|y - \bar{y}\|.$$

Since $\int_{t_0}^{\infty} \lambda(t)\, dt < 1$, F is a contraction mapping on $S(\varepsilon)$. hence by the well known fixed point theorem the system (2) has a unique solution $y \in S(\varepsilon)$ with $\|y\| < \varepsilon$. This completes the proof of this theorem.

REMARK. Theorem 1 may be regarded as a stability result for the system (2) in the following sense.

For every $\varepsilon > 0$ and sufficiently small such that for every solution $x(t)$ of (1) with $\|x\| < \varepsilon$, there exists a solution y of (2) with $\|y\| < \varepsilon$.

THEOREM 2. *Suppose that $x = 0$ of (1) is globably uniformly stable in variation, and that $g(t, y, z)$ satisfies the inequality*

$$|g(s, y, z)| \leqq h(s)(|y(s)| + |z(s)|),$$

where $h \in C[I, R_+]$ and $\int_{t_0}^{\infty} h(s)\, ds < \infty$, Further suppose that the operator T satisfies the inequality

$$|Ty(t)| \leqq \int_{t_0}^{t} q(s)|y(s)|^p\, ds,$$

where $q \in C[I, R_+]$, $\int_{t_0}^{\infty} q(s)\, ds < \infty$ and $p \geqq 1$. Then the solution $y = 0$ of (2) is stable.

Proof. Since the solution $x = 0$ of (1) is globally uniformly stable in variation, it follows from Lemma 3 with $x_0 = 0$ that

$$
\begin{aligned}
|y(t, t_0, y_0)| &\\
\leqq |y_0| \sup_{m\in D} |\phi(t, t_0, m)| &\\
+ \int_{t_0}^{t} |\phi(t, s, y(s, t_0, y_0))|\,|g(s, y(s, t_0, y_0),\ Ty(s, t_0, y_0))|\,ds &\\
\leqq M|y_0| + M\int_{t_0}^{t} h(s)\,|y(s, t_0, y_0)|\,ds &\\
+ M\int_{t_0}^{t} h(s)\Big(\int_{t_0}^{s} q(u)\,|y(u, t_0, y_0)|^{p}\,du\Big)\,ds\,. &
\end{aligned}
$$

If $p=1$, then from Lemma 4, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$
\begin{aligned}
&|y(t, t_0, y_0)|\\
&\quad\leqq M|y_0|\Big[1+M\int_{t_0}^{t} h(s)\\
&\qquad\cdot\exp\Big(\int_{t_0}^{s}(Mh(u)+q(u))\,du\Big)ds\Big]<\varepsilon,\\
&\qquad\qquad \textit{for}\quad t\geqq t_0\geq 0,\quad\text{and}\quad |y_0|<\delta\,.
\end{aligned}
$$

If $p>1$, since $\int_{t_0}^{\infty} h(s)\,ds<\infty$, $\int_{t_0}^{\infty} q(s)\,ds<\infty$, then for every $\varepsilon>0$, there exists $\delta>0$ such that

$$
\begin{aligned}
1>1+(1-p)(M\delta)^{p-1}\int_{t_0}^{\infty} q(v)&\\
\cdot\exp\Big((p-1)\,M\int_{t_0}^{v} h(u)\,du\Big)\,dv<\frac{1}{2}\,,&
\end{aligned}
$$

and

$$
M\delta\Big\{1+\Big[\exp\,(M\int_{t_0}^{\infty} h(s)\,ds)-1\Big]\Big(\frac{1}{2}\Big)^{1/1-p}\Big\}<\varepsilon\,.
$$

It follows from Lemma 6, we have

$$
\begin{aligned}
|y(t, t_0, y_0)| &\leqq M\,|y_0|\Big\{1+\int_{t_0}^{t} Mh(s)\\
&\qquad\cdot\exp\Big(M\int_{t_0}^{s} h(u)\,du\Big)\cdot\Big[1+(1-p)(M|y_0|)^{p-1}\\
&\qquad\cdot\int_{t_0}^{s} q(v)\exp\Big((p-1)M\int_{t_0}^{v} h(u)\,du\Big)dv\Big]^{1/1-p}ds\Big\}\\
&\leqq M\delta\Big\{1+\int_{t_0}^{t} Mh(s)\\
&\qquad\exp\Big(M\int_{t_0}^{s} h(u)\,du\Big)\cdot\Big[1+(1-p)(M|\delta|)^{p-1}\\
&\qquad\cdot\int_{t_0}^{s} q(v)\exp\Big((p-1)M\int_{t_0}^{v} h(u)\,du\Big)dv\Big]^{1/1-p}ds\\
&\leq M|\delta|\Big\{1+\Big[\exp\Big(\int_{t_0}^{\infty} Mh(s)ds\Big)-1\Big]\Big(\frac{1}{2}\Big)^{1/1-p}<\varepsilon,\\
&\qquad\qquad\qquad\text{for } |y_0|<\delta\,.
\end{aligned}
$$

This shows that the solution $y=0$ of (2) is stable

THEOREM 3. *Let the function* $g(t, y, Ty)$ *satisfy an inequality*

$$\begin{aligned}|\phi(t,s,y(s))g(s,y(s),\ Ty(s)) - \phi(t,s,\bar{y}(s))g(s,\bar{y}(s),\ T\bar{y}(s))| \\ \leqq h(s)|y(s)-\bar{y}(s)| + \int_{t_0}^{s} q(u)|y(u)-\bar{y}(u)|^{p}\,du\,,\end{aligned} \tag{12}$$

where $h \in C[I,\ R_+]$, $\int_{t_0}^{\infty} h(s)\,ds < \infty$, $q \in C[I,\ R_+]$, $\int_{t_0}^{\infty} q(s)\,ds < \infty$ *and* $p > 1$.

Suppose further that system (1) is stable. Then there exists two constants C and δ having the following property: For any two solution $\bar{y}(t, t_0, \bar{y}_0)$ and $y(t, t_0, y_0)$ of system (2) such that $|y_0 - \bar{y}_0| < \delta$ and $t \geqq t_0 \geqq 0$, we have

$$|y(t,\ t_0,\ y_0) - \bar{y}(t,\ t_0,\ \bar{y}_0)| < C\delta\,.$$

Proof. Assume that the system (1) is stable, then for any $\delta > 0$,

$$|y_0 - \bar{y}_0| < \delta$$

implies

$$|x(t,\ t_0,\ y_0) - \bar{x}(t,\ t_0,\ \bar{y}_0)| < C_1\,\delta\,.$$

Since $p > 1$, $\int_{t_0}^{\infty} h(s)\,ds < \infty$, and $\int_{t_0}^{\infty} q(s)\,ds < \infty$, there exists a $\delta > 0$ such that

$$1 > 1 + (1-p)(C_1\,\delta)^{p-1} \int_{t_0}^{\infty} q(s)\exp\left((p-1)\int_{t_0}^{s} h(u)\,du\right)ds > \frac{1}{2}\,.$$

It follows from Lemmas 1 and 6, we have

$$\begin{aligned}&|y(t,\ t_0,\ y_0) - \bar{y}(t.\ t_0,\ \bar{y}_0)| \\ &\quad\leq |\,x(t,\ t_0,\ y_0) - \bar{x}(t,\ t_0,\ \bar{y}_0)| \\ &\qquad + \int_{t_0}^{t} |\phi(t,\ s,\ y(s,\ t_0,\ y_0))\,g(s,\ y(s,\ t_0,\ y_0),\ Ty(s,\ t_0,\ y_0)) \\ &\qquad - \phi(t,\ s,\ \bar{y}(t,\ s,\ \bar{y}_0))\,g(s,\ \bar{y}(s,\ t_0,\ \bar{y}_0),\ T\bar{y}(s,\ t_0,\ \bar{y}_0))|\,ds \\ &\quad\leqq C_1\,\delta + \int_{t_0}^{t} h(s)|\,y(s,\ t_0,\ y_0) - \bar{y}(s,\ t_0,\ \bar{y}_0)|\,ds \\ &\qquad + \int_{t_0}^{t} h(s)\int_{t_0}^{s} q(u)|\,y(u,\ t_0,\ y_0) - \bar{y}(u,\ t_0,\ \bar{y}_0)|^{p}\,du\,ds \\ &\quad\leq C_1\,\delta + \int_{t_0}^{t} h(s)\exp\left(\int_{t_0}^{s} h(u)\,du\right)\cdot\Big[(C_1\,\delta)^{1-p} \\ &\qquad + (1-p)\int_{t_0}^{s} q(u)\exp\left((p-1)\int_{t_0}^{u} h(w)\,dw\right)du\Big]^{1/1-p} ds\end{aligned}$$

$$\leqq C_1\delta\Big\{1+\int_{t_0}^{t} h(s)\exp\Big(\int_{t_0}^{s} h(u)\,du\Big)\,ds$$
$$\cdot\Big[1+(1-p)(C_1\delta)^{p-1}\int_{t_0}^{\infty} q(u)$$
$$\cdot\exp\Big((p-1)\int_{t_0}^{u} h(w)\,dw\Big)\,du\Big]^{1/1-p}\Big\}$$
$$\leqq C_1\delta\Big\{1+\Big[\exp\Big(\int_{t_0}^{t} h(s)\,ds\Big)-1\Big]\Big(\frac{1}{2}\Big)^{1/1-p}\Big\}$$
$$\leqq C_1\delta\Big\{1+\Big[\exp\Big(\int_{t_0}^{\infty} h(s)\,ds\Big)-1\Big]\Big(\frac{1}{2}\Big)^{1/1-p}\Big\}=C\delta,$$
$$\text{for } t\geqq t_0,\ |y-\bar{y}_0|<\delta.$$

REMARK. In this theorem, althrough condition (12) is not so general as the condition (3.1) of Theorem 1 [15], the condition that the stability of equation (3.2) of Theorem 1 [5] is deleted in this theorem, and we still have the similar result.

THEOREM 4. *Let the function $g(t, y, Ty)$ satisfy an inequality*

$$|\phi(t,s,y(s))\,g(s,y(s),Ty(s))-\phi(t,s,\bar{y}(s),T\bar{y}(s))g(s,\bar{y}(s),T\bar{y}(s))|$$
$$\leq h(s)(|y(s)-\bar{y}(s)|+\int_{t_0}^{s} q(u)|y(u)-\bar{y}(u)|\,du$$

where $h\in C[I, R_+]$, $\int_{t_0}^{\infty} h(s)\,ds<\infty$, $q\in C[I, R_+]$, $\int_{t_0}^{\infty} q(s)\,ds<\infty$.

Then the stability of system (2) follows from system (1).

Proof. Assume that system (1) is stable, then for every $\delta>0$, there exists $C_1>0$, such that

$$|x(t, t_0, y_0)-\bar{x}(t, t_0, \bar{y}_0)|<C_1\delta,$$

whenever $t\geqq t_0\geq 0$, and $|y_0-\bar{y}_0|<\delta$. It follows from Lemmas 1 and 4, we have

$$|y(t, t_0, y_0)-\bar{y}(t, t_0, \bar{y}_0)|$$
$$\leqq |x(t, t_0, y_0)-\bar{x}(t, t_0, \bar{y}_0)|$$
$$+\int_{t_0}^{t} |\phi(t, s, y(s, t_0, y_0))g(s, y(s, t_0, y_0)), Ty(s, t_0, y_0))$$
$$-\phi(t, s, \bar{y}(s, t_0, \bar{y}_0))\,g(s, \bar{y}(s, t_0, \bar{y}_0), T\bar{y}(s, t_0, \bar{y}_0))|\,ds$$
$$\leqq C_1\delta+\int_{t_0}^{t} h(s)|y(s, t_0, y_0)-\bar{y}(s, t_0, \bar{y}_0)|\,ds$$
$$+\int_{t_0}^{t} h(s)\int_{t_0}^{s} q(v)|y(v, t_0, y_0)-\bar{y}(v, t_0, \bar{y}_0)|\,dv\,ds$$
$$\leqq C_1\delta\Big[1+\Big(\int_{t_0}^{t} h(u)\cdot\exp\Big(\int_{t_0}^{u}(h(s)+q(s))\,ds\Big)du\Big)\Big],$$
$$\text{for } t\geqq t_0\geqq 0 \text{ and } |y_0-\bar{y}_0|<\delta.$$

This shows that the stability of system (2).

THEOREM 5. *Suppose that the solution* $x=0$ *of* (1) *is exponentially asymptotically stable and that*

$$g(t, y, Ty) = g_1(t, y) + g_2(t, y) + g_3(t, y) \tag{13}$$

where g_1, g_2, *and* $g_3 \in C[I \times R^n, R^n]$

$$g_1(t, y) = o(|y|) \text{ as } |y| \to 0 \text{ uniformly in } t,$$

and $|g_2(t, y)| \leqq h(t)[|y| + \exp(-C_1(t-t_0)) \int_{t_0}^t q(s)|y(s)|^p]ds)$ *where* h, q, p, *are the same as defined in Theorem* 3 *and* $C_1 > 0$. *Suppose further that there exists* $A > 0$ *such that*

$$|g_3(t, y)| \leqq \lambda(t) \quad (t \geqq 0, |y| \leqq A),$$

where $\lambda \in C[I, R_+]$ *and*

$$\Lambda(t) = \int_t^{t+1} \lambda(s)\, ds \longrightarrow 0 \text{ as } t \to \infty.$$

Then there exists $T > 0$ *such that every solution* $y(t)$ *of* (2) *for which* $|y(t_0)|$ *is sufficiently small for any* $t_0 \geqq T$ *tends to zero as* $t \to \infty$.

Proof. We first show that the solution $z=0$ of the equation

$$z' = f(t, z) + g_2(t, z) \tag{14}$$

is exponentially asymptotically stable. Following the similar argument as in the proof of Theorems 1 and 2 given in [4], we can show in a suitable region, we have

$$|\phi(t, t_0, z_0)| \leqq K_1 \exp(-C_1(t-t_0)), \quad t \geqq t_0 \geqq 0,\ K_1 > 0 \text{ and } C_1 > 0.$$

It follows from Lemma 3 with $x_0 = 0$, we have

$$\begin{aligned} &|z(t, t_0, z_0)| \\ &\quad \leqq |z_0| \sup_{m \in D} |\phi(t, t_0, m)| \\ &\qquad + \int_{t_0}^t |\phi(t, s, z(s, t_0, z_0))|\,|g_2(s, z(s, t_0, z_0))|\, ds \\ &\quad \leqq K_1 |z_0| \exp(-C_1(t-t_0)) \\ &\qquad + \int_{t_0}^t K_1 \exp(-C_1(t-s)) h(s) |z(s, t_0, z_0)|\, ds \\ &\qquad + K_1 \int_{t_0}^t \exp(-C_1(t-s)) h(s) \\ &\qquad\quad \cdot \exp(-C_1(s-t_0)) \int_{t_0}^s q(u) |z(u, t_0, z_0)|^p\, du\, ds, \\ &\hspace{20em} \text{for } t \geqq t_0 \geq 0. \end{aligned}$$

Hence

$$\begin{aligned}|z(t, t_0, z_0)| \exp(C_1(t-t_0)) \\ = K_1|z_0| + K_1 \int_{t_0}^{t} h(s) \exp(C_1(s-t_0))\, |z(s, t_0, z_0)|\, ds \\ + K_1 \int_{t_0}^{t} h(s) \int_{t_0}^{s} q(u) \exp(-pc_1(u-t_0))|\, z(u, t_0, z_0) \\ \cdot \exp(C_1(u-t_0))|^{p}\, du\, ds, \ (t \geqq t_0 \geqq 0).\end{aligned}$$

Since $p>1$, $\int_{t_0}^{\infty} h(s)\, ds < \infty$, $\int_{t_0}^{\infty} q(s)\, ds < \infty$, then there exists $\delta > 0$ such that

$$\begin{aligned}1 > 1 + (1-p)(K_1\delta)^{p-1} \int_{t_0}^{\infty} q(s) \\ \cdot \exp(-pC_1(s-t_0)) \exp\left((p-1)K_1 \int_{t_0}^{s} h(u)\, du\right) ds > \frac{1}{2}.\end{aligned}$$

Then it it follows from Lemma 6 and the fact .$p>1$, we have

$$\begin{aligned}&|z(t, t_0, z_0)| \exp(C_1(t-t_0)\\ &\leqq K_1|z_0| + K_1 \int_{t_0}^{t} h(s) \exp\left(K_1 \int_{t_0}^{s} h(u)\, du\right) \\ &\quad \cdot \Big[(K_1|z_0|)^{1-p} + (1-p) \int_{t_0}^{s} q(v)(-pC_1(v-t_0)) \\ &\quad \cdot \exp\left((p-1) K_1 \int_{t_0}^{v} h(u)\, du\right) dv\Big]^{1/1-p} ds. \\ &\leqq K_1\, |z_0| \Big\{1 + K_1 \int_{t_0}^{t} h(s) \exp\left(K_1 \int_{t_0}^{s} h(u)\, du\right) ds \\ &\quad \cdot \Big[1 + (K_1\delta)^{p-1}(1-p) \int_{t_0}^{\infty} q(v) \\ &\quad \cdot \exp\left((p-1)K_1 \int_{t_0}^{v} h(u)\, du\right) dv\Big]^{1/1-p}\Big\} \\ &\leqq K_1\, |z_0| \left\{1 + \left[K_1 \int_{t_0}^{\infty} h(s) \exp\left(K_1 \int_{t_0}^{s} h(u)\, du\right) ds\right]\left(\frac{1}{2}\right)^{1/1-p}\right\} \\ &\leqq K_1\, |z_0| \left\{1 + \left[\exp\left(K_1 \int_{t_0}^{\infty} h(s)\, ds\right) - 1\right] \cdot \left(\frac{1}{2}\right)^{1/1-p}\right\}, \\ &\qquad\qquad \text{for } |z_0| < \delta,\ t \geqq t_0 \geqq 0.\end{aligned}$$

Hence

$$\begin{aligned}|z(t, t_0, z_0)| \leqq 2K_1\delta \exp(-C_1(t-t_0)) \\ \cdot \exp\left(K_1 \int_{t_0}^{\infty} h(s)\, ds\right)\left(\frac{1}{2}\right)^{1/1-p}, \\ \text{for } |z_0| < \delta.\end{aligned}$$

This shows that the solution $z=0$ of the equation (14) is exponentially asymptotically stable. Again we may regard (14) as an unperturbed system and regard the equation (13) as a perturbed system, and the Theorem follows immediately from Lemma 5.

REMARK. (5. A). If $q=0$, this Theorem is still true.

(5. B). For the case $g_1 = g_3 = 0$ and $p=1$ is due to Pachpatte [15].

(5. C). For the case $g_1 = g_3 = 0$ and $0 \leqq p < 1$ is due to Pachatte [16].

(5. D). If $q=0$, $f(t, y) = A\,y$, where A is a constant matrix and all the characteristic roots of A have negative real parts, Then Theorem 5 is reduced to Theorem 3.2 [18].

(5. E). If $q=0$ and $g_3 = 0$, then Theorem 5 is reduced to Corollary of Theorem 2 [4].

(5. F). Using Remark (5. C) and Lemma 5, we can show that Theorem 5 is still true if the condition $p>1$ is replaced by $0 \leqq p < 1$.

(5. G). Using Remark (5. B) and Lemma 5, we can show that Theorem 5 is still true if the condition $p>1$ is replaced by the condition $p=1$.

F. Brauer. [5] shows that if the solution $x=0$ is globally uniformly stable, and if perturbation $g(t, y)$ satisfies

$$|g(t, y)| \leqq h(t)|y|, \qquad t \geqq 0, \ |y| < \infty,$$

where h satisfies an inequality of the form

$$h(t) \leqq \frac{k}{t}$$

for large $t>0$. Then the solutions of (2) do not grow more rapidly than polynomials as $t \to \infty$.

If we apply Lemma 6, then following similar argument as in [5], we can extend this result to obtain Theorem 6.

THEOREM 6. *Let the solution $x=0$ of (1) be globally uniformly stable in variation. Suppose that the perturbation $g(t, y, z)$ satisfies*

$$|g(t, y, z)| \leqq h(t)[|y| + |z|]$$

where h satisfies an inequality of the form

$$h(t) \leqq \frac{k}{t}$$

for large $t>0$, *and* $h \in C[I, R_+]$. *Suppose that the operator* T *satisfies the inequality of the form*

$$|Ty(t)| \leqq \int_{t_0}^{t} q(s)|y(s)|^p \, ds,$$

where $q \in C[I, R_+]$, $\int_{t_0}^{\infty} q(s)\, ds < \infty$, *and* p *is a constant,* $0 \leqq p < 1$, *then any solution* $y(t, t_0, y_0)$ *of perturbed system* (2) *does not grow more rapidly than polynomials as* $t \to \infty$.

Proof. It follows from Lemma 3 with $x_0 = 0$ that

$$\begin{aligned} |y(t, t_0, y_0)| &\leqq |y_0| \sup_{m \in D} |\phi(t, t_0, m)| \\ &\quad + \int_{t_0}^{t} |\phi(t, s, y(s, t_0, y_0))|\,|g(s, y(s, t_0, y_0))|\, ds \\ &\leqq M|y_0| + M \int_{t_0}^{t} h(s)|y(s, t_0, y_0)|\, ds \\ &\quad + M \int_{t_0}^{t} h(s) \int_{t_0}^{s} q(v)|y(v, t_0, y_0)|^p \, dv\, ds, \end{aligned}$$

then it follows from Lemma 6, we have

$$\begin{aligned} &|y(t, t_0, y_0)| \\ &\leqq M|y_0| + M \int_{t_0}^{t} h(s) \exp\left(M \int_{t_0}^{s} h(u)\, du\right) \\ &\quad \cdot \left[(M|y_0|)^{1-p} + (1-p) \int_{t_0}^{s} q(v)\right. \\ &\quad \cdot \left.\exp\left((p-1) M \int_{t_0}^{v} h(u)\, du\right) dv\right]^{1/1-p} ds \\ &\leqq M|y_0| \left[1 + M \int_{t_0}^{t} h(s) \exp\left(M \int_{t_0}^{s} h(u)\, du\right)\right. \\ &\quad \cdot \left[1 + (1-p)(M|y_0|)^{p-1} \left(\int_{t_0}^{\infty} q(v)\right.\right. \\ &\quad \cdot \left.\left.\left.\exp\left((p-1) M \int_{t_0}^{v} h(u)\, du\right) dv\right)^{1/1-p}\right] ds\right] \\ &\leqq M\, |y_0| \left\{1 + \left[\exp\left(M \int_{t_0}^{t} h(s)\, ds\right) - 1\right]\right. \\ &\quad \cdot \left.\left[1 + (1-p)(M|y_0|)^{p-1} \int_{t_0}^{\infty} q(v)\, dv\right]^{1/1-p}\right\} \\ &\leqq 2M|y_0| \exp\left(M \int_{t_0}^{t} h(s)\, ds\right) \\ &\quad \cdot \left[1 + (1-p)(M|y_0|)^{p-1} \int_{t_0}^{\infty} q(s)\, ds\right]^{1/1-p} \\ &= \frac{2M|y_0|}{t_0^{MK}} t^{MK} \left[1 + (1-p)(M|y_0|)^{p-1} \int_{t_0}^{\infty} q(s)\, ds\right]^{1/1-p} \\ &= \frac{2}{t_0^{MK}} t^{MK} \left[M|y_0|^{1-p} + (1-p) \int_{t_0}^{\infty} q(s)\, ds\right]^{1/1-p}, \\ &\qquad\qquad \text{for} \quad |y_0| < \infty. \end{aligned}$$

which proves the result.

REMARK. In Theorem 6, if the condition $0 \leq p < 1$ is replaced by $p > 1$ and suppose that there exist two positive constants M and k such that

$$\int_{t_0}^{\infty} q(s) \exp\left((p-1)M \int_{t_0}^{s} h(v)\, dv\right) ds < k,$$

then with the similar argument as in the proof of Theorem 6, we can show that there exists a positive constant δ such that for $|y_0| < \delta$, and solution $y(t, t_0, y_0)$ of (2) does not grow more rapidly than polynomials as $t \to \infty$.

THEOREM 7. *Let the fundamental matrix $\phi(t, t_0, y_0)$ of the variational system be uniformly slowly growing and let the perturbation $g(t, y, z)$ satisfy*

$$|g(t, y, z)| \leqq h(t)[|y| + |Z|],\ t > 0,\ |y| < \infty.$$

where $(1/t)\int_0^t h(s)\, ds \to 0$ as $t \to \infty$. Suppose further that the operator T satisfies the inequality

$$|Ty(t)| \leqq \exp(\varepsilon t) \int_{t_0}^{t} q(s)|y(s)|^p ds, \qquad t \geqq t_0 \geqq 0,\ |y| < \infty,$$

where $0 \leqq p < 1$, $h, q \in C[I, R_+]$ and $\varepsilon > 0$, suppose that there exists $C > 0$ such that

$$\int_{t_0}^{\infty} q(s) \exp(\varepsilon ps) < C.$$

Then any solution $y(t, t_0, y_0)$ of the perturbed system (2) is slowly growing.

Proof. Fix $t_0 \geqq 0$, ε and y_0. Then by the definition of uniformly slowing growing, there exists $K = K(\varepsilon)$ such that for all $t \geqq t_0$, we have

$$|\phi(t, t_0, y_0)| \leqq K \exp(\varepsilon(t - t_0)).$$

Then it follows from Lemma 3 with $x_0 = 0$ that

$$
\begin{aligned}
&|y(t, t_0, y_0)| \\
&\quad \leqq |y_0| \sup_{m \in D} |\phi(t, t_0, m)| \\
&\qquad + \int_{t_0}^{t} |\phi(t, s, y(s, t_0, y_0))| \, |g(s, y(s, t_0, y_0))| \, ds \\
&\quad \leqq K|y_0| \exp(\varepsilon\, t) + \int_{t_0}^{t} K \exp(\varepsilon(t-s))\, h(s) |y(s, t_0, y_0)| \, ds \\
&\qquad + \int_{t_0}^{t} K \exp(\varepsilon(t-s))\, h(s) \\
&\qquad \cdot \exp(\varepsilon s) \int_{t_0}^{s} q(u) |\, y(u, t_0, y_0)|^p \, du \, ds \\
&\quad \leqq K|y_0| \exp(\varepsilon\, t) + K \int_{t_0}^{t} \exp(\varepsilon\, t) h(s) |\exp(-\varepsilon\, s) y(s, t_0, y_0)| \, ds \\
&\qquad + K \int_{t_0}^{t} \exp(\varepsilon\, t)\, h(s) \int_{t_0}^{s} q(u) \\
&\qquad \cdot \exp(\varepsilon\, pu) |y(u, t_0, y_0) \exp(-\varepsilon\, u)|^p \, du \, ds .
\end{aligned}
$$

From this and Lemma 6 and the fact $0 \leqq p < 1$, we have

$$
\begin{aligned}
&y(t, t_0, y_0)(-\varepsilon\, t) \\
&\quad = K|y_0| + K \int_{t_0}^{t} h(s) |y(s, t_0, y_0)| \exp(-\varepsilon\, s)\, ds \\
&\qquad + K \int_{t_0}^{t} h(s) \int_{t_0}^{s} q(u) \exp(\varepsilon\, pu) |y(u, t_0, y_0) \\
&\qquad \cdot \exp(-\varepsilon u)|^p \, du \, ds \\
&\quad \leqq K|y_0| \Big\{ 1 + K \int_{t_0}^{t} h(s) \\
&\qquad \cdot \exp\Big(K \int_{t_0}^{s} h(v)\, dv \Big) \cdot \Big[1 + (1-p)(K|y_0|)^{p-1} \\
&\qquad \cdot \int_{t_0}^{s} q(v) \exp(\varepsilon\, pv) \exp\Big((p-1) K \int_{t_0}^{v} h(u)\, du \Big) dv \Big]^{1/1-p} ds \Big\} \\
&\quad \leqq K|y_0| \Big\{ 1 + \Big[K \int_{t_0}^{t} h(s) \exp\Big(K \int_{t_0}^{s} h(v)\, dv \Big) ds \Big] \\
&\qquad \cdot \Big[1 + (1-p)(K|y_0|)^{p-1} \int_{t_0}^{\infty} q(v) \\
&\qquad \cdot \exp(\varepsilon\, pv) \exp\Big((p-1) K \int_{t_0}^{v} h(u)\, du \Big) dv \Big]^{1/1-p} \Big\} \\
&\quad \leqq K|y_0| \Big\{ 1 + \Big[\exp\Big(K \int_{t_0}^{t} h(s)\, ds \Big) - 1 \Big] \\
&\qquad \cdot \Big[1 + (1-p)(K|y_0|)^{p-1} \int_{t_0}^{\infty} q(s) \exp(\varepsilon\, ps)\, ds \Big] \Big\} \\
&\quad \leqq 2K|y_0| \Big\{ \exp\Big(K \int_{t_0}^{t} h(s) ds \Big) \\
&\qquad \cdot \Big[1 + (1-p)(K|y_0|)^{p-1} \int_{t_0}^{\infty} q(s) \exp(\varepsilon\, ps)\, ds \Big] \Big\} .
\end{aligned}
$$

Let $\Lambda(t) = (1/t)\int_{t_0}^{t} h(s)\,ds$, and let $T = T(\varepsilon)$ be so large that $K\Lambda(t) < \varepsilon$ for all $t \geqq T$. Then

$$\begin{aligned}
&|y(t, t_0, y_0)| \exp(-\varepsilon t) \\
&\quad \leqq 2K|y_0| \exp(Kt\Lambda(t)) \\
&\qquad \cdot\left[1 + (1-p)(K|y_0|)^{p-1}\int_{t_0}^{\infty} q(v)\exp(\varepsilon pv)\,dv\right]^{1/1-p} \\
&\quad \leqq \exp(\varepsilon t)\, 2K|y_0| \\
&\qquad \cdot\left[1 + (1-p)(K|y_0|)^{p-1}\int_{t_0}^{\infty} q(v)\exp(\varepsilon pv)\,dv\right]^{1/1-p}, \\
&\quad \leqq \exp(\varepsilon t)\, 2K|y_0|\,[1 + (1-p)(K|y_0|)^{p-1}C]^{1/1-p}, \\
&\qquad\qquad \text{for } t \geq t_0 \geq 0,\ |y_0| < \infty.
\end{aligned}$$

Thus

$$|y(t, t_0, y_0)| \leqq 2\exp(2\varepsilon t)[(K|y_0|)^{1-p} + (1-p)C]^{1/1-p}, \quad \text{for } t \geq t_0 \geq 0,\ |y_0| < \infty.$$

This show that each solution $y(t, t_0, y_0)$ of (2) is slowly growing.

REMARK. If we use the similar conditions and follow the similar argument as given in Theorem 7, we can show that there exists a positive constant δ such that any solution $y(t, t_0, y_0)$ of the perturbed system (2) is slowly growing, whenever $|y_0| < \delta$ and $p > 1$.

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