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Moreau–Rockafellar Type Theorem for Convex Set Functions*

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Let (X, Γ, μ) be an atomless finite measure space and $\mathscr{S} \subset \Gamma$ a convex subfamily. It is proved that the Moreau–Rockafellar theorem, $\partial(F_1 + \cdots + F_n)(\Omega) = \partial F_1(\Omega) + \cdots + \partial F_n(\Omega)$, holds for proper convex set functions $F_1, ..., F_n$ and $\Omega \in \mathscr{S}$ if all set functions F_i, except possibly one, are w^*-lower semicontinuous on $\mathscr{S}$. As applications, the Kuhn–Tucker type condition for an optimal solution of convex programming problem with set functions and the Fritz John type condition for an optimal solution of vector-valued minimization problem for set functions are obtained.

1. INTRODUCTION

Throughout the following let (X, Γ, μ) be a finite atomless measure space and $F_1, F_2, ..., F_n$, $G_1, G_2, ..., G_m$ be convex real-valued set functions defined on a convex subfamily $\mathscr{S}$ of the σ-field Γ. We consider an optimization problem as follows:

(P) Minimize: $F(\Omega) = (F_1(\Omega), F_2(\Omega), ..., F_n(\Omega))$

Subject to: $\Omega \in \mathscr{S}$ and $G_j(\Omega) \leqslant 0$ $j = 1, 2, ..., m$.

Because the linear operations can not be applied to σ-field Γ, the convexity of set functions must be first defined. This type of problems has many interesting applications in fluid flow, electrical insulator design, and

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optimal plasma confinement (see the references in [13], see also [2, 3, 8, 9]). In [13], Morris introduced the notions of differentiability and convexity of a real valued set function on a measure space. Following Morris setting, Lai *et al.* proved the Fenchel duality theorem for set functions [8] and characterized an optimal solution for a minimization problem of convex set functions in terms of the saddle point of a Lagrangian function [9].

Recently, Chou, Hsia, and Lee have studied the programming problems for set functions in [2, 3]. In [2], they used the Farkas–Minkowski, theorem to establish a necessary condition for the optimality of convex set functions with a constraint qualification; and in [3], they considered the second-order differentiable set functions and proved a second-order necessary condition for a local minimum of a minimization problem with an inequality constraint for set functions.

In this paper we will prove a theorem of Moreau–Rockafellar type for set functions, and then use the theorem to prove a Kuhn–Tucker type condition for an optimal solution of the minimization problem (P) for real valued set functions. If the set functions are vector-valued, the Fritz John type condition for an optimum of the multiobjective minimization problem (P) is established. The Kuhn–Tucker type condition for an optimal solution of functions on the usual linear space has been shown in Mond and Zlobec [12, Theorem 2] as well as in Kanniappan and Sastry [7, Theorem 2.2], while the Fritz John type condition has been proved in Lai and Ho [10, Theorem 3.1].

2. Definitions and Basic Properties for Set Functions

We assume that (X, Γ, μ) is an atomless finite measure space. Each $\Omega \in \Gamma$ can be identified with its characteristic function $\chi_\Omega \in L_\infty(X, \Gamma, \mu) \subset L_1(X, \Gamma, \mu)$ and so the σ-field Γ is identified as a subset $\chi_\Gamma = \{\chi_\Omega \mid \Omega \in \Gamma\}$ of $L_\infty(X, \Gamma, \mu) = L^\infty$. For a convex set function $F\colon \mathscr{S} \to \mathbb{R}$, we admit $F(\Omega) = F(\Lambda)$ if $\chi_\Omega = \chi_\Lambda$, μ-a.e., thus F can be regarded as a function defined on $\chi_{\mathscr{S}} = \{\chi_\Omega : \Omega \in \mathscr{S}\}$ in L^∞. Similar to [13, Proposition 3.2 and Lemma 3.3], for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exist sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Omega \setminus \Lambda} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda)\, \chi_{\Lambda \setminus \Omega} \tag{1}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Omega + (1-\lambda)\, \chi_\Lambda, \tag{2}$$

where w^* stands for the weak* convergence (cf. Morris [13]). The sequene $\{V_n = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ satisfying (1) and (2) is called *Morris sequence* associated with $(\Omega, \Lambda, \lambda)$.

DEFINITION 1. A subfamily $\mathscr{S}$ of Γ is called *convex* if any $(\Omega, \Lambda, \lambda) \in \mathscr{S} \times \mathscr{S} \times [0, 1]$ associated with a Morris sequence $\{V_n\}$ in Γ exists a subsequence $\{V_{n_k}\}$ such that

$$V_{n_k} = \Omega_{n_k} \cup \Lambda_{n_k} \cup \{\Omega \cap \Lambda\} \in \mathscr{S} \qquad \text{for all } k. \tag{3}$$

DEFINITION 2. A set function $F: \mathscr{S} \to \mathbb{R}$ is called *convex* on a convex subfamily $\mathscr{S} \subset \Gamma$ if for any $(\Omega, \Lambda, \lambda) \in \mathscr{S} \times \mathscr{S} \times [0, 1]$, there exists a Morris sequence $\{V_n\}$ in $\mathscr{S}$ such that

$$\overline{\lim}\, F(V_n) \leqslant \lambda F(\Omega) + (1 - \lambda)\, F(\Lambda). \tag{4}$$

DEFINITION 3. A subset $B \subset \mathbb{R} \times \Gamma$ is called *convex* if for any (r, Ω), $(s, \Lambda) \in B$, and $\lambda \in [0, 1]$ and any Morris sequence $\{V_n\}$ associated with $(\Omega, \Lambda, \lambda)$, there exist a subsequence $\{V_{n_k}\}$ of $\{V_n\}$ and a sequence $t_k \to \lambda r + (1 - \lambda)\, s$ such that $\{(t_k, V_{n_k})\} \subset B$.

DEFINITION 4. Let $F: \Gamma \to \mathbb{R}^{\#} = \mathbb{R} \cup \{\infty\}$ be a set function with

$$\operatorname{Dom} F = \{\Omega \in \Gamma \mid \Gamma(\Omega) \text{ is finite}\} \equiv \mathscr{S}.$$

(i) F is called *w^*-lower* (resp. *w^*-upper*) *semicontinuous* (w^*-l.s.c./w^*-u.s.c.) at $\Omega \in \mathscr{S}$ if $-\infty < F(\Omega) \leqslant \underline{\lim} F(\Omega_n)$ (resp. $\infty > F(\Omega) \geqslant \overline{\lim}\, F(\Omega_n)$) for any sequence $\Omega_n \in \mathscr{S}$ with $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$.

(ii) F is called *w^*-continuous* at $\Omega \in \mathscr{S}$ if $F(\Omega) = \lim F(\Omega_n)$ for any sequence $\Omega_n \in \mathscr{S}$ with $\chi_{\Omega_n} \to^{w^*} \chi_\Omega$.

We will assume $F(\varnothing) = 0$ throughout.

PROPOSITION 1. *Any convex set function F on a convex family $\mathscr{S} \subset \Gamma$ is w^*-upper semicontinuous.*

Proof. Take $\Lambda = \varnothing$, $\Lambda_n = \varnothing$, and $\lambda = 1$ in (1) and (2). Then for any $\Omega \in \mathscr{S}$, there is a sequence $\{\Omega_n\} \subset \Gamma$ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \chi_\Omega = \chi_{\Omega \setminus \varnothing}.$$

It follows that

$$\begin{aligned} \overline{\lim}\, F(\Omega_n) &= \overline{\lim}\, F(\Omega_n \cup \Lambda_n \cup (\Omega \cap \varnothing)) \\ &\leqslant F(\Omega) + (1 - 1)\, F(\varnothing) \\ &= F(\Omega). \end{aligned}$$

Hence F is w^*-upper semicontinuous on $\mathscr{S}$. Q.E.D.

The following corollary follows immediately from Proposition 1.

COROLLARY 2. *Every w^*-lower semicontinuous convex set function is w^*-continuous.*

Let $\bar{\Gamma}$ denote the w^*-closure of χ_Γ in L^∞. Then $\bar{\Gamma}=\{f\in L^\infty \mid 0\leqslant f\leqslant 1\}$ (cf. [4, Corollary 3.6]). If $A\subset \mathbb{R}\times\Gamma$, we use $\bar{A}$ to denote the w^*-closure of A in $\mathbb{R}\times L^\infty$.

Let $\mathcal{N}(f)$ be the family of all w^*-neighborhoods of $f\in\bar{\Gamma}$. We now extend a convex set function F on a convex subfamily $\mathcal{S}$ to its w^*-closure $\bar{\mathcal{S}}$.

DEFINITION 5. The *w^*-lower* (resp. *w^*-upper*) *semicontinuous hull* of a set function F on $\mathcal{S}\subset\Gamma$ is a functional $\bar{F}$ (resp. $\hat{F}$) on $\bar{\mathcal{S}}$ defined by

$$\bar{F}(f)=\sup_{V\in\mathcal{N}(f)}\ \inf_{\Omega\in V\cap\mathcal{S}} F(\Omega) \qquad \text{for} \qquad f\in\bar{\mathcal{S}} \tag{5}$$

$$(\text{resp. } \hat{F}(f)=\inf_{V\in\mathcal{N}(f)}\ \sup_{\Omega\in V\cap\mathcal{S}} F(\Omega) \text{ for } f\in\bar{\mathcal{S}}).$$

The following proposition follows immediately from Definitions 4 and 5.

PROPOSITION 3. (i) *$\bar{F}(\Omega)\leqslant F(\Omega)\leqslant\hat{F}(\Omega)$ for all $\Omega\in\mathcal{S}$.*

(ii) *If F is w^*-l.s.c. (resp. w^*-u.s.c.), then $F(\Omega)=\bar{F}(\Omega)$ (resp. $F(\Omega)=\hat{F}(\Omega)$) for $\Omega\in\mathcal{S}$.*

(iii) *If F is w^*-continuous on $\mathcal{S}$, then $\bar{F}=\hat{F}$ on $\bar{\mathcal{S}}$. It follows that $\bar{F}$ is the unique w^*-continuous extension of F.*

(iv) *If F is convex on a convex subfamily $\mathcal{S}$, then $\bar{\mathcal{S}}$ is convex in L^∞ and $\bar{F}$ is convex on $\bar{\mathcal{S}}$ (cf.* [4, Corollary 3.10]).

For a convex set function $F\colon\mathcal{S}\to\mathbb{R}$ on convex subfamily $\mathcal{S}$ we set

$$[F,\mathcal{S}]=\{(r,\Omega)\in\mathbb{R}\times\Gamma \mid \Omega\in\mathcal{S},\ F(\Omega)\leqslant r\}.$$

Then $[F,\mathcal{S}]$ is a convex family of $\mathbb{R}\times\Gamma$. It follows immediately from [4, Proposition 3.9 and Corollary 3.10] that

LEMMA 4. *Let $\mathcal{S}\to\mathbb{R}$ be a convex set function on the convex family $\mathcal{S}\subset\Gamma$. Then*

$$\overline{[F,\mathcal{S}]}=[\bar{F},\bar{\mathcal{S}}], \tag{6}$$

and $[\bar{F},\bar{\mathcal{S}}]$ is a convex subset of $\mathbb{R}\times L^\infty$.

LEMMA 5 (cf. [4, Corollary 3.12]). *Let $F\colon\mathcal{S}\to\mathbb{R}$ be a convex w^*-continuous set function. If $\bar{\mathcal{S}}$ has a relative interior point* (w.r.t. *the L^∞-norm topology*), *then $[\bar{F},\bar{\mathcal{S}}]$ has a relative interior point.*

DEFINITION 6 (cf. [9, Definition 1]). The element $f \in L_1(X, \Gamma, \mu)$ is called a *subgradient* of a convex set function F at $\Omega_0 \in \Gamma$ if it satisfies the inequality

$$F(\Omega) \geqslant F(\Omega_0) + \langle \chi_\Omega - \chi_{\Omega_0}, f \rangle \qquad \text{for all} \qquad \Omega \in \Gamma. \tag{7}$$

The set of all subgradients of a set function F at Ω_0 is denoted by $\partial F(\Omega_0)$ and is called the *subdifferential* of f at Ω_0. If $\partial F(\Omega_0) \neq \varnothing$, then F is called *subdifferentiable* at Ω_0.

It is easy to show that a point Ω^* minimizes $F(\Omega)$ for $\Omega \in \Gamma$ if and only if $0 \in \partial F(\Omega^*)$.

3. THE MOREAU–ROCKAFELLAR THEOREM

A function g from a Banach space V to $\mathbb{R} \cup \{\infty\}$ is called *proper* if g does not take $-\infty$ and does not identically equal to ∞.

The Moreau–Rockafellar theorem can be stated in its general form

MOREAU–ROCKAFELLAR THEOREM. *Let $g_1, ..., g_n$ be proper convex real-valued functions on a Banach space V. Then*

$$\partial g_1(x) + \cdots + \partial g_n(x) \subset \partial(g_1 + \cdots + g_n)(x)$$

for every $x \in V$. If all functions $g_1, ..., g_n$, except possibly one, are continuous at a point $x_0 \in (\mathrm{Dom}\, f_1) \cap \cdots \cap (\mathrm{Dom}\, f_n)$, then

$$\partial g_1(x) + \cdots + \partial g_n(x) = \partial(g_1 + \cdots + g_n)(x)$$

for all $x \in V$.

This theorem plays an important role in the theory of optimization for nondifferentiable convex functions. We say that a set function $F: \Gamma \to \mathbb{R} \cup \{\infty\}$ is *proper* if $F \not\equiv \infty$ on Γ. The following is a theorem of Moreau–Rockafellar type for convex set functions.

THEOREM 6. *Let $F_1, F_2: \Gamma \to \mathbb{R} \cup \{\infty\}$ be proper convex set functions on* $\mathrm{Dom}\, F_1 = \mathrm{Dom}\, F_2 = \mathscr{S}$. *Then*

$$\partial F_1(\Omega) + \partial F_2(\Omega) \subset \partial(F_1 + F_2)(\Omega) \qquad \textit{for all} \quad \Omega \in \Gamma. \tag{8}$$

Suppose that $\mathscr{S}$ is a convex subfamily of Γ and that $\bar{\mathscr{S}}$, the weak-closure of $\mathscr{S}$, has a relative interior point, if F_1 is w^*-continuous on $\mathscr{S}$, then*

$$\partial(F_1 + F_2)(\Omega) = \partial F_1(\Omega) + \partial F_2(\Omega) \qquad \textit{for all} \qquad \Omega \in \Gamma. \tag{9}$$

Proof. The inclusion (8) follows immediately from the definition of subdifferential of set functions.

We prove only the equality (9). For $\Omega \in \mathscr{S}$ and $f \in \partial(F_1 + F_2)(\Omega)$, we define

$$G_1(\Lambda) = F_1(\Lambda) - F_1(\Omega) - \langle \chi_\Lambda - \chi_\Omega, f \rangle$$

and

$$G_2(\Lambda) = F_2(\Lambda) - F_2(\Omega) \qquad \text{for} \qquad \Lambda \in \Gamma.$$

Since F_1 and F_2 are proper convex set functions, G_1 and G_2 are proper convex set functions on $\mathscr{S}$, and

$$G_1(\Omega) = G_2(\Omega) = 0 = (G_1 + G_2)(\Omega).$$

As $f \in \partial(F_1 + F_2)(\Omega)$ we have

$$\begin{aligned}(G_1 + G_2)(\Lambda) - 0 &= (F_1 + F_2)(\Lambda) - (F_1 + F_2)(\Omega) - \langle \chi_\Lambda - \chi_\Omega, f \rangle \\ &\geqslant 0 \qquad \text{for all } \Lambda \in \Gamma,\end{aligned}$$

it follows that $0 \in \partial(G_1 \in G_2)(\Omega)$ and

$$\min_{\Lambda \in \Gamma} (G_1 + G_2)(\Lambda) = G_1(\Omega) + G_2(\Omega) = 0. \tag{10}$$

Let $C_1 = [\bar{G}_1, \bar{\mathscr{S}}]$ and $C_2 = \{(\gamma, h): (-\gamma, h) \in [\bar{G}_2, \bar{\mathscr{S}}]\}$. Then from Lemma 4, C_1 and C_2 are convex subsets of $\mathbb{R} \times L^\infty(X, \Gamma, \mu)$. Since F_1 is w^*-continuous on $\mathscr{S}$ and $\bar{\mathscr{S}}$ contains a relative interior point, it follows from Lemma 5 that C_1 has a relative interior point. In order to apply the separation theorem, we need to prove that $(\operatorname{ri} C_1) \cap C_2 = \varnothing$, where $\operatorname{ri} C_1$ denotes the relative interior points of C_1. If not, let $(\gamma, h) \in (\operatorname{ri} C_1) \cap C_2$. Then there exists an $\varepsilon > 0$ such that $\bar{G}_1(h) < \gamma - \varepsilon$ and a sequence $\{\Omega_n\}$ in $\mathscr{S}$ such that $\chi_{\Omega_n} \to^{w^*} h$ and $\underline{\lim} G_2(\Omega_n) \leqslant -\gamma$. Since $(\gamma, h) \in C_2$, we have

$$(-\gamma, h) \in [\bar{G}_2, \bar{\mathscr{S}}] = [\overline{G_2, \mathscr{S}}].$$

Since $\bar{G}_1$ is w^*-continuous on $\bar{\mathscr{S}}$, $\lim_{n \to \infty} G_1(\Omega_n) = \bar{G}_1(h)$. Hence there is a sufficiently large n such that

$$G_1(\Omega_n) < \gamma - \varepsilon \qquad \text{and} \qquad G_2(\Omega_n) < -\gamma + \varepsilon$$

which implies

$$(G_1 + G_2)(\Omega_n) < 0.$$

This contradicts (10). Hence

$$C_2 \cap (\operatorname{ri} C_1) = \varnothing.$$

Thus C_1 and C_2 can be properly separated by a hyperplane in $\mathbb{R} \times L^{\infty}$. Since $[G_1, \mathscr{S}] \subset C_1$ and the set $B \equiv \{(\gamma, \Omega): (-\gamma, \Omega) \in [G_2, \mathscr{S}]\} \subset C_2$, this hyperplane can separate $[G_1, \mathscr{S}]$ and B. By assumption ri $\bar{\mathscr{S}} \neq \varnothing$, the hyperplane is not vertical. Thus the nonzero functional can be taken by $(-1, g) \in \mathbb{R} \times L_1(X, \Gamma, \mu)$ such that

$$\sup_{(\gamma, f) \in C_1} \langle (\gamma, h), (-1, g) \rangle \leqslant \inf_{(\gamma, h) \in C_2} \langle (\gamma, h), (-1, g) \rangle.$$

That is, there exists an $\alpha \in \mathbb{R}$ such that

$$\sup_{(\gamma, \chi_\Lambda) \in [G_1, \mathscr{S}]} \{\langle \chi_\Lambda, g \rangle - \gamma\} \leqslant \alpha \leqslant \inf_{(\gamma, \chi_\Lambda) \in B} \{\langle \chi_\Lambda, g \rangle - \gamma\}.$$

Since $(G_1(\Omega), \chi_\Omega) = (-G_2(\Omega), \chi_\Omega) = (0, \chi_\Omega)$ belongs to $[G_1, \mathscr{S}] \cap B$, it follows that

$$\langle \chi_\Lambda, g \rangle - G_1(\Lambda) \leqslant \alpha = \langle \chi_\Omega, g \rangle - G_1(\Omega)$$

for all $\Lambda \in \mathscr{S}$ and

$$\langle \chi_\Lambda, g \rangle + G_2(\Lambda) \geqslant \alpha = \langle \chi_\Omega, g \rangle + G_2(\Omega)$$

for all $\Lambda \in \mathscr{S}$. In other words,

$$G_1(\Lambda) \geqslant G_1(\Omega) + \langle \chi_\Lambda - \chi_\Omega, g \rangle$$

and

$$G_2(\Lambda) \geqslant G_2(\Omega) + \langle \chi_\Lambda - \chi_\Omega, -g \rangle \qquad \text{for all} \qquad \Lambda \in \mathscr{S}.$$

Since G_1 and G_2 are proper convex set function, thus for any $\Lambda \notin \mathscr{S}$, $G_1(\Lambda) = \infty$, and $G_2(\Lambda) = \infty$. Hence

$$G_1(\Lambda) \geqslant G_1(\Omega) + \langle \chi_\Lambda - \chi_\Omega, g \rangle,$$
$$G_2(\Lambda) \geqslant G_2(\Omega) + \langle \chi_\Lambda - \chi_\Omega, -g \rangle \qquad \text{for all } \Lambda \in \Gamma;$$

that is, $g \in \partial G_1(\Omega)$ and $-g \in \partial G_2(\Omega)$, so it follows that

$$0 \in \partial G_1(\Omega) + \partial G_2(\Omega) = \partial F_1(\Omega) - f + \partial F_2(\Omega).$$

Consequently,

$$f \in \partial F_1(\Omega) + \partial F_2(\Omega).$$

Therefore,

$$\partial(F_1 + F_2)(\Omega) \subset \partial F_1(\Omega) + \partial F_2(\Omega) \qquad \text{for } \Omega \in \mathscr{S}.$$

If $\Omega \notin \mathscr{S}$, then $F_1(\Omega) + F_2(\Omega) = \infty$ and $\partial(F_1 + F_2)(\Omega) = \varnothing$. Thus

$$\partial(F_1 + F_2)(\Omega) \subset \partial F_1(\Omega) + \partial F_2(\Omega) \qquad \text{for} \qquad \Omega \in \Gamma. \tag{11}$$

From (8) and (11), we obtain (9). The proof is complete. Q.E.D.

Remark. According to Corollary 2, the condition of w^*-continuous in Theorem 6 can be replaced by w^*-lower semicontinuous.

The following corollary follows immediately from Theorem 6.

COROLLARY 7. *Let $F_1, F_2, ..., F_n: \Gamma \to \mathbb{R} \cup \{\infty\}$ be proper convex set functions on $\mathscr{S} = \operatorname{Dom} F_i$, $i = 1, 2, ..., n$. Then*

$$\partial F_1(\Omega) + \cdots + \partial F_n(\Omega) \subset \partial(F_1 + \cdots + F_n)(\Omega)$$

for all $\Omega \in \Gamma$. Suppose that $\mathscr{S}$ is a convex subfamily of Γ, $\bar{\mathscr{S}}$ contains a relative interior point and all functions F_i, except possibly one, are w^*-continuous on $\mathscr{S}$, then

$$\partial(F_1 + \cdots + F_n)(\Omega) = \partial F_1(\Omega) + \cdots + \partial F_n(\Omega) \tag{12}$$

for all $\Omega \in \Gamma$.

In Proposition 3(iii), we have already proved that a w^*-continuous convex set function F on a convex subfamily $\mathscr{S}$ has a unique w^*-continuous extension $\bar{F}$. We will show that the Moreau–Rockafellar theorem holds for functions $\bar{F}$. At first we show a relation between the subdifferentials of F and $\bar{F}$.

LEMMA 8. *Let $\mathscr{S}$ be a convex subfamily of Γ and $F: \Gamma \to \mathbb{R} \cup \{\infty\}$ be w^*-continuous and convex on $\mathscr{S}$. We assume further that $\bar{F}$ is the w^*-continuous extension of F to $\bar{\mathscr{S}}$. Then*

$$\partial F(\Omega) = \partial \bar{F}(\Omega) \qquad \textit{for all} \qquad \Omega \in \mathscr{S}.$$

Proof. Let $\Omega \in \mathscr{S}$ and $g \in \partial \bar{F}(\Omega)$. Then

$$\bar{F}(f) \geqq \bar{F}(\Omega) + \langle f - \chi_\Omega, g \rangle \qquad \text{for all} \qquad f \in L^\infty.$$

Since $\bar{F}(\Lambda) = F(\Lambda)$ for $\Lambda \in \mathscr{S}$ (see Proposition 3(ii)), we have

$$F(\Lambda) \geqslant F(\Omega) + \langle \chi_\Lambda - \chi_\Omega, g \rangle \qquad \text{for} \qquad \Lambda \in \mathscr{S}.$$

If $\Lambda \notin \mathscr{S}$, then $F(\Lambda) = \infty$. Thus

$$F(\Lambda) \geqslant F(\Omega) + \langle \chi_\Lambda - \chi_\Omega, g \rangle \qquad \text{for all} \qquad \Lambda \in \Gamma.$$

This shows that $g \in \partial F(\Omega)$ and $\partial \bar{F}(\Omega) \subset \partial F(\Omega)$.

Conversely, if $g \in \partial F(\Omega)$ then for any $f \in \bar{\mathscr{S}}$ corresponding to a sequence $\{\Omega_n\} \subset \mathscr{S}$ that $\chi_{\Omega_n} \to^{w^*} f$ implies

$$\bar{F}(f) = \lim_{n \to \infty} \bar{F}(\Omega_n) \geqslant \lim_{n \to \infty} [\bar{F}(\Omega) + \langle \chi_{\Omega_n} - x_\Omega, g \rangle$$

$$= \bar{F}(\Omega) + \langle f - \chi_\Omega, g \rangle,$$

and if $f \in L^\infty \backslash \bar{\mathscr{S}}$ then $\bar{F}(f) = \infty$, so

$$\bar{F}(f) \geqslant \bar{F}(\Omega) + \langle g, f - \chi_\Omega \rangle \qquad \text{for all} \qquad f \in L^\infty.$$

This shows that $g \in \partial \bar{F}(\Omega)$. Hence $\partial F(\Omega) = \partial \bar{F}(\Omega)$ for $\Omega \in \mathscr{S}$. Q.E.D.

THEOREM 9. *In Theorem* 6, *if both* F_1 *and* F_2 *are* w^**-continuous on* $\mathscr{S}$, *then*

(i) $\partial(\bar{F}_1 + \bar{F}_2)(f) = \partial \bar{F}_1(f) + \partial \bar{F}_2(f)$ *for* $f \in L^\infty$,

(ii) $\partial(F_1 + F_2)(\Omega) = \partial F_1(\Omega) + \partial F_2(\Omega)$ *for* $\Omega \in \mathscr{S}$.

Proof. Since $\bar{F}_1$ and $\bar{F}_2$ are w^*-continuous on the w^*-compact set $\bar{\mathscr{S}}$, (i) follows from the Moreau–Rockafellar theorem in Banach space and (ii) follows from Lemma 8. Q.E.D.

4. KUHN–TUCKER TYPE CONDITION FOR SET FUNCTIONS

Let $F, G_1, G_2, ..., G_m$ be real-valued set functions on Γ. We consider, in this section, a single objective optimization problem for set functions in the following form

(P_1) Minimize: $F(\Omega)$
Subject to: $\Omega \in \mathscr{S}$ and $G_j(\Omega) \leqslant 0$, $j = 1, 2, ..., m$,
where $\mathscr{S}$ is a subfamily of Γ.

The main purpose of this section is to show that a necessary condition of Kuhn–Tucker type holds for an optimal solution of problem (P_1) for set functions. We need the following lemma (cf. [2, Theorem 3.2]).

LEMMA 10. *In problem* (P_1), *let* $F, G_1, ..., G_n$ *be real-valued convex set functions on a convex family* $\mathscr{S} \subset \Gamma$. *We assume further the Slater condition*: *there exists a set* $\Omega_0 \in \mathscr{S}$ *such that* $G_j(\Omega_0) < 0$, $j = 1, 2, ..., m$. *If* $\Omega^* \in \mathscr{S}$ *is an optimal solution of* (P_1), *then there exist nonnegative real numbers* $\lambda_1^*, ..., \lambda_m^*$ *with* $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$, *such that*

$$\sum_{j=1}^{m} \lambda_j^* G_j(\Omega^*) \equiv \langle \lambda^*, G(\Omega^*) \rangle = 0, \tag{13}$$

and (Ω^*, λ^*) *is a saddle point of the Lagrangian function* $L(\Omega, \lambda) = F(\Omega) + \langle \lambda, G(\Omega) \rangle$. *That is,*

$$F(\Omega^*) + \langle \lambda, G(\Omega^*) \rangle \leqslant F(\Omega^*) + \langle \lambda^*, G(\Omega^*) \rangle$$
$$\leqslant F(\Omega) + \langle \lambda^*, G(\Omega) \rangle \tag{14}$$

for all $\lambda = (\lambda_1, ..., \lambda_m)$ *with* $\lambda_i \geqslant 0$ *and* $\Omega \in \mathscr{S}$.

THEOREM 11. *Let* $F, G_1, ..., G_m$ *in* (P_1) *be proper convex set functions on a convex family* $\mathscr{S} \subset \Gamma$ *and satisfy the Slater's condition* (*cf. Lemma* 10). *We assume further that all of the set functions* $F, G_1, ..., G_m$, *except possibly one, are* w^**-lower semicontinuous on* $\mathscr{S}$ *and that* $\bar{\mathscr{S}}$ *contains a relative interior point. If* $\Omega^* \in \mathscr{S}$ *is a solution to* (P_1), *then there exists* $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$ *with* $\lambda_j^* \geqslant 0$, *such that*

$$\text{(i)} \qquad \langle \lambda^*, G(\Omega^*) \rangle = 0 \tag{15}$$

and

$$\text{(ii)} \qquad 0 \in \partial F(\Omega^*) + \sum_{j=1}^{m} \lambda_j^* \, \partial G_j(\Omega^*) + N_{\mathscr{S}}(\Omega^*) \tag{16}$$

where

$$N_{\mathscr{S}}(\Omega^*) = \{ f \in L_1(X, \Gamma, \mu) \mid \langle \chi_\Omega - \chi_{\Omega^*}, f \rangle \leqslant 0 \text{ for all } \Omega \in \mathscr{S} \}.$$

Proof. Let

$$\Phi_{\mathscr{S}}(\Omega) = \begin{cases} 0 & \text{if } \Omega \in \mathscr{S} \\ +\infty & \text{if } \Omega \notin \mathscr{S}. \end{cases}$$

Then $\Phi_{\mathscr{S}}$ is clearly a convex proper set function on Γ and w^*-continuous on $\mathscr{S}$. Let $\Omega^* \in \mathscr{S}$ be an optimal solution of (P_1). It follows from Lemma 10 that there exists $\lambda^* = (\lambda_1^*, ..., \lambda_m^*)$ with $\lambda_j^* \geqslant 0$ such that

$$\langle \lambda^*, G(\Omega^*) \rangle = 0,$$

and (λ^*, Ω^*) is a saddle point of the Lagrangian $L(\Omega, \lambda) = F(\Omega) + \langle \lambda, G(\Omega) \rangle$. Thus, by definition of $\Phi_{\mathscr{S}}$,

$$F(\Omega^*) + \langle \lambda^*, G(\Omega^*)) \rangle + \Phi_{\mathscr{S}}(\Omega^*) \leqslant F(\Omega) + \langle \lambda^*, G(\Omega) \rangle + \Phi_{\mathscr{S}}(\Omega)$$

for all $\Omega \in \Gamma$, and so

$$F(\Omega^*) + \langle \lambda^*, G(\Omega^*) \rangle + \Phi_{\mathscr{S}}(\Omega^*) = \inf_{\Omega \in \Gamma} [F(\Omega) + \langle \lambda^*, G(\Omega) \rangle + \Phi_{\mathscr{S}}(\Omega)].$$

Therefore

$$0 \in \partial\left(F + \sum_{j=1}^{m} \lambda_j^* G_j + \Phi_{\mathscr{S}}\right)(\Omega^*).$$

By Corollary 7, we obtain

$$\begin{aligned} 0 &\in \partial F(\Omega^*) + \sum_{j=1}^{m} \lambda_j^* \, \partial G_j(\Omega^*) + \partial \Phi_{\mathscr{S}}(\Omega^*) \\ &= \partial F(\Omega^*) + \sum_{j=1}^{m} \lambda_j^* \, \partial G_j(\Omega^*) + N_{\mathscr{S}}(\Omega^*), \end{aligned}$$

where

$$\begin{aligned} N_{\mathscr{S}}(\Omega^*) &= \partial \Phi_{\mathscr{S}}(\Omega^*) \\ &= \{ f \in L_1(X, \Gamma, \mu) \mid \langle \chi_\Omega - \chi_{\Omega^*}, f \rangle \leqslant 0 \text{ for all } \Omega \in \mathscr{S} \}. \end{aligned}$$

Q.E.D.

5. Fritz John Type Condition for Vector-Valued Minimization for Set Functions

In this section, we consider the vector-valued minimization problem for set functions in the following form

(P) Minimize: $F(\Omega) = (F_1(\Omega), ..., F_n(\Omega))$
Subject to: $\Omega \in \mathscr{S}$ and $G_j(\Omega) \leqslant 0$, $j = 1, ..., m$, where $F_i: \mathscr{S} \to \mathbb{R}$, $i = 1, 2, ..., n$, $G_j: \mathscr{S} \to \mathbb{R}$, $j = 1, 2, ..., m$, and $\mathscr{S} \subset \Gamma$.

For $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ in $\mathbb{R}^n$, we use the notations

$$\begin{array}{llll} x < y & \text{if } x_i < y_i & \text{for each } i = 1, 2, ..., n; & \\ x \leqslant y & \text{if } x_i \leqslant y_i & \text{for each } i = 1, 2, ..., n & \text{and } x \neq y; \\ x \leqq y & \text{if } x_i \leqslant y_i & \text{for each } i = 1, 2, ..., n. & \end{array}$$

We say that a set $\Omega^* \in \mathscr{S} \subset \Gamma$ is a *Pareto optimal* solution of the vector-valued set function $F: \mathscr{S} \to \mathbb{R}^n$ if there is no $\Omega \in \mathscr{S}$ such that $F(\Omega) \leqslant F(\Omega^*)$.
A necessary condition for the existence of an optimal solution of the optimization problem (P) will be given in this section. It is a Fritz John type condition (cf. Lai and Ho [10]) which we state in the following theorem.

THEOREM 12. *In problem* (P), *let* $\mathscr{S}$ *be a convex subfamily of* Γ *and* F_i, $i = 1, 2, ..., n$, G_j, $j = 1, 2, ..., m$, *be proper convex set functions on* Γ. *Let* Ω_0

be a Pareto optimal solution of problem (P). *Suppose that for each* $i \in \{1, 2, ...,\}$ *there corresponds a* $\Omega_i \in \mathscr{S}$ *such that*

$$\begin{aligned} &G_k(\Omega_i) < 0, && k = 1, 2, ..., m \\ &F_j(\Omega_i) < F_j(\Omega_0) && \text{for} \quad j = 1, 2, ..., n, j \neq i \end{aligned} \tag{17}$$

and that all functions $F_1, ..., F_n, G_1, ..., G_m$, *except possibly one, are w*-continuous on* $\mathscr{S}$ *and that* $\bar{\mathscr{S}}$ *contains a relative interior point, then there exist* $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ *with* $\alpha_i \geqslant 1$, $i = 1, 2, ..., n$, *and* $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ *in* $\mathbb{R}^m_+$ *such that*

(i) $\displaystyle\sum_{k=1}^{m} \lambda_k G_k(\Omega_0) = 0$

(ii) $\displaystyle 0 \in \sum_{j=1}^{n} \alpha_j\, \partial F_j(\Omega_0) + \sum_{k=1}^{m} \lambda_k\, \partial G_k(\Omega_0) + N_{\mathscr{S}}(\Omega_0).$

To prove this theorem we need the following lemma in vector minimization for set functions which is similar to Lemma 3.1 of [6] for usual vector minimization problem (cf. also [10]).

LEMMA 13. *Let* $\mathscr{S}$ *be a convex subfamily of* Γ *and* $F_1, ..., F_n$ *be proper convex set functions on* Γ *with domain* $\mathscr{S}$. *Then the problem* (P) *has an optimal solution* (*in Pareto sense*) *at* $\Omega_0 \in \mathscr{S}$ *if and only if* Ω_0 *minimizes each* F_j *on the constraint set*

$$C_j = \{\Omega \in \mathscr{S}\colon F_i(\Omega) \leqslant F_i(\Omega_0),\ i \neq j,\ G(\Omega) \leqq 0\} \tag{18}$$

where $G(\Omega) = (G_1(\Omega), ..., G_m(\Omega))$, $j = 1, 2, ..., n$.

The proof of this lemma follows from the argument used in [6, Lemma 3.1].

Proof of Theorem 12. Let Ω_0 be a Pareto optimal solution of (P). By Lemma 13, Ω_0 minimizes each F_i, $i = 1, 2, ..., n$, on the constraint set C_i of (18). Then, in view of Theorem 11, there exist

$$\alpha^{(i)} = (\alpha_{1i}, ..., \alpha_{ni}) \in \mathbb{R}^n_+ \qquad \text{and} \qquad \beta^{(i)} = (\beta_{1i}, ..., \beta_{mi}) \in \mathbb{R}^m_+$$

with $\alpha_{ii} = 1$ such that

$$0 \in \sum_{j=1}^{n} \alpha_{ji}\, \partial F_j(\Omega_0) + \sum_{k=1}^{m} \beta_{ki}\, \partial G_k(\Omega_0) + N_{\mathscr{S}}(\Omega_0) \tag{19}$$

and

$$\sum_{k=1}^{m} \beta_{ki} G_k(\Omega_0) = 0, \qquad i = 1, 2, ..., n. \tag{20}$$

Let $i = 1, 2, ..., n$, in (19) and then sum them up; we obtain

$$0 \in \left(1 + \sum_{i=2}^{n} \alpha_{1i}\right) \partial F_1(\Omega_0) + \cdots + \left(\sum_{i=1}^{n-1} \alpha_{ni} + 1\right) \partial F_n(\Omega_0)$$

$$+ \sum_{k=1}^{m} (\beta_{k1} + \cdots + \beta_{kn}) \, \partial G_k(\Omega_0) + n N_{\mathscr{S}}(\Omega_0)$$

$$= \sum_{j=1}^{n} \alpha_j \, \partial F_j(\Omega_0) + \sum_{k=1}^{m} \lambda_k \, \partial G_k(\Omega_0) + N_{\mathscr{S}}(\Omega_0),$$

where $\alpha_j = \alpha_{j1} + \cdots + \alpha_{j,\, j-1} + 1 + \alpha_{j,\, j+1} + \cdots + \alpha_{jn} \geqslant 1$,

$$\lambda_k = \sum_{i=1}^{n} \beta_{ki} \geqslant 0, \qquad k = 1, 2, ..., m,$$

and

$$\sum_{1}^{m} \lambda_k G_k(\Omega_0) = \sum_{k=1}^{m} \sum_{i=1}^{n} \beta_{ki} G_k(\Omega_0) = 0.$$

This proves the theorem. Q.E.D.

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