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Optimality of Differentiable, Vector-Valued n-Set Functions*

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The necessary and sufficient conditions for the existence of an optimal solution of a vector-valued, n-set functions optimization problem is obtained in this paper.

1. INTRODUCTION

Let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable and $F: \mathscr{S} \to \mathbb{R}^m$, $G: \mathscr{S} \to \mathbb{R}^p$ defined on a convex subfamily $\mathscr{S}$ of $\Gamma^n = \Gamma \times \cdots \times \Gamma$, we consider an optimization problem as

$$\text{minimize } F(\Omega_1, ..., \Omega_n) \text{ subject to } (\Omega_1, ..., \Omega_n) \in \mathscr{S}$$
$$\text{and } G(\Omega_1, ..., \Omega_n) \leqq 0. \tag{P}$$

In [12], Morris first considered the general theory of real-valued set functions of a single set. He showed the necessary and sufficient conditions for a constrained local minimum of real-valued set functions of a single set. Following the Morris setting, Chou *et al.* [1] characterized the proper efficient solutions for the problem (P) in terms of a optimal solution for associated scalar problems. In [13], Tanaka considered the Pareto optimization of (P) and showed the necessary and sufficient conditions for the existence of the local Pareto minimum to (P). In [1, 6, 7, 12, 13], the optimization problem has remained confined to set functions of a single set. In [4], Corley first developed the general theory for n-set functions and gave the concepts of partial derivative and derivative of n-set function. In this paper, we prove the Farkas–Minkowski type theorem for vector-valued n-set functions. Using this result we establish the necessary and suf-

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ficient conditions for the existence of weak local minimum to (P) in terms of the derivatives of vector-valued n-set functions involved. Because the Pareto minimum to (P) is also the weak minimum, but the converse is not true, hence our results and methods are quite different from Theorems 1 and 2 of [13]. When the objective functions are real-valued, our results reduce to Theorems 3.7, 3.8, and 4.7 of [4].

2. Preliminary

Throughout this paper, we assume (X, Γ, μ) is a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable and let $\Gamma^n = \Gamma \times \cdots \times \Gamma = \{(\Omega_1, ..., \Omega_n) \mid \Omega_i \in \Gamma,\ i=1, ..., n\}$. We define a pseudometric d on Γ^n as

$$d[(\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)] = \left\{ \sum_{i=1}^{n} [\mu(\Omega_i \Delta \Lambda_i)]^2 \right\}^{1/2},$$

$\Omega_i, \Lambda_i \in \Gamma$, $i=1, ..., n$, where $\Omega_i \Delta \Lambda_i$ denotes symmetric difference for Ω_i and Λ_i. Essentially $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n)$ will be regarded as equivalent if $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) = 0$. We see that Γ^n is only a semialgebra but not a σ-algebra. For $f \in L_1(X, \Gamma, \mu)$ and $\Omega \in \Gamma$, the integral $\int_\Omega f\, d\mu$ will be denoted by $\langle f, \chi_\Omega \rangle$, where χ_Ω denotes the characteristic function of Ω. We introduce the following notations for the vectors in the m-dimensional Euclidean space $\mathbb{R}^m$. For two vectors $x = (x_1, ..., x_m)$ and $y = (y_1, ..., y_m)$ in $\mathbb{R}^m$,

(i) $x < y$ iff $x_i < y_i$ for all $i = 1, ..., m$.

(ii) $x \leq y$ iff $x_i \leq y_i$ for all $i = 1, ..., m$ and $x \neq y$.

(iii) $x \leqq y$ iff $x_i \leq y_i$ for all $i = i, ..., m$.

The zero vector $(0, ..., 0)$ in $\mathbb{R}^m$ is denoted by 0 and the nonnegative orthant is denoted by $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m \mid x \geqq 0\}$. We denote by $B(\mathbb{R}^m, \mathbb{R}^p)$, the set of all continuous linear operators from $\mathbb{R}^m$ to $\mathbb{R}^p$ and

$$B^+(\mathbb{R}^m, \mathbb{R}^p) = \{w \in B(\mathbb{R}^m, \mathbb{R}^p) \mid w(\mathbb{R}^m_+) \subset \mathbb{R}^p_+\}.$$

Definition 2.1. Let $A \subset \mathbb{R}^m$, a point $y_0 \in A$ is said to be a weak minimum of A, denoted by $y_0 \in$ w-min A if there does not exist y in A such that $y < y_0$, and $y_0 \in A$ is said to be a minimum of A if $y_0 \leqq y$ for all $y \in A$.

Definition 2.2. A set function $F: \Gamma \to \mathbb{R}$ is differentiable at $\Omega \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$, the derivative of F at Ω such that

$$F(\Lambda) = F(\Omega) + \langle f, \chi_\Lambda - \chi_\Omega \rangle + d(\Omega, \Lambda)\, E(\Omega, \Lambda),$$

where

$$\lim_{d(\Omega, \Lambda) \to 0} E(\Omega, \Lambda) = 0.$$

DEFINITION 2.3. Let $F: \Gamma^n \to \mathbb{R}$ and $(\Omega_1, ..., \Omega_n) \in \Gamma^n$. Then F is said to have a partial derivative at $(\Omega_1, ..., \Omega_n)$ with respect to Λ_i if the set function

$$H(\Lambda_i) = F(\Omega_1, ..., \Omega_{i-1}, \Lambda_i, \Omega_{i+1}, ..., \Omega_n)$$

has derivative h_{Ω_i} at Ω_i. In this case we define the ith partial derivative of F at $(\Omega_1, ..., \Omega_n)$ to be $f^i_{\Omega_1, ..., \Omega_n} = h_{\Omega_i}$.

Now, we define the derivative of vector-valued n-set functions.

DEFINITION 2.4. Let $\mathscr{S} \subset \Gamma^n$, $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ and $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$. Then F is said to be differentiable at $(\Omega_1, ..., \Omega_n)$ if the partials $f^{ij}_{\Omega_1, ..., \Omega_n}$, $i = 1, 2, ..., n$, of F_j exist for each $j = 1, 2, ..., m$ and satisfy

$$\begin{aligned} F(\Lambda_1, ..., \Lambda_n) &= F(\Omega_1, ..., \Omega_n) \\ &\quad + \left(\sum_{i=1}^{n} \langle f^{i1}_{\Omega_1, ..., \Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{im}_{\Omega_1, ..., \Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\ &\quad + W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)), \quad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}, \end{aligned}$$

where

$$\frac{W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))}{d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))} \to 0$$

as $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) \to 0$.

Throughout the paper if $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ and $G = (G_1, ..., G_p)$: $\mathscr{S} \to \mathbb{R}^p$ are differentiable at $(\Omega_1, ..., \Omega_n)$, we will denote $f^{ij}_*, ..., g^{ij}_*$ the ith partial derivatives of F_j and G_j at $(\Omega_1, ..., \Omega_n)$, respectively.

Similar to [12, Proposition 3.2 and Lemma 3.3], for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exists sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Lambda \setminus \Omega} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1 - \lambda) \chi_{\Omega \setminus \Lambda} \tag{1}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_{\Lambda} + (1 - \lambda) \chi_{\Omega}, \tag{2}$$

where w^* stands for the w^*-convergence. The sequence $\{V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ satisfying (1) and (2) is called the Morris sequence associated with $(\Omega, \Lambda, \lambda)$.

DEFINITION 2.5. A subfamily $\mathscr{S}$ of Γ^n is convex if given $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$, where N is the set of natural numbers.

DEFINITION 2.6. A set function $F: \mathscr{S} \to \mathbb{R}^m$ is called $\mathbb{R}_+^m$-convex on a convex subfamily $\mathscr{S}$ of Γ^n if for each $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1 - \lambda) F(\Omega_1, ..., \Omega_n).$$

EXAMPLE. If $F: \Gamma^n \to \mathbb{R}^m$ is convex on Γ^n, then the subfamily

$$\mathscr{S} = \{(\Omega_1, ..., \Omega_n) \in \Gamma^n \mid F(\Omega_1, ..., \Omega_n) < 0\}$$

is a convex subfamily of Γ^n.

3. MAIN RESULTS

DEFINITION 3.1. Let $\mathscr{S}$ be a nonempty subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$. Then $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$ if $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$ if there exists $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ satisfying $d[(\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)] < \delta$.

THEOREM 1. *Let $\mathscr{S}$ be a convex subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$ be a $\mathbb{R}_+^m$-convex set function. If $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$, then $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$.*

Proof. Since $(\Omega_1, ..., \Omega_n)$ is a local minimum of F on $\mathscr{S}$, there exists $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) \leqq F(\Lambda_1, ..., \Lambda_n)$ for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ with $d[(\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)] < \delta$. Fix $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$. Then by the convexity of F on the convex subfamily $\mathscr{S}$ of Γ^n, for any $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1 - \lambda) F(\Omega_1, ..., \Omega_n).$$

Since

$$\begin{aligned}\lim_{k\to\infty} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) &= \lim_{k\to\infty} \left\{\sum_{k=1}^{n} [\mu(V_i^k(\lambda)\, \Delta\Omega_i)]^2\right\} \\ &= \lim_{k\to\infty} \left\{\sum_{i=1}^{n} \|\chi_{V_i^k(\lambda)} - \chi_{\Omega_i}\|_{L_1}^2\right\}^{1/2} \\ &= \left\{\sum_{i=1}^{n} \lambda^2 \|\chi_{A_i} - \chi_{\Omega_i}\|_{L_1}^2\right\}^{1/2} \\ &= \lambda\left\{\sum_{i=1}^{n} [\mu(A_i\, \Delta\Omega_i)]^2\right\}^{1/2} \\ &= \lambda d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)),\end{aligned}$$

there exists $r > 0$ and a natural number M such that

$$d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < \delta \quad \text{for} \quad 0 < \lambda < r \quad \text{and} \quad k \geq M.$$

Hence

$$F(\Omega_1, ..., \Omega_n) \leqq F(V_1^k(\lambda), ..., V_n^k(\lambda)) \quad \text{for} \quad 0 < \lambda < r \quad \text{and} \quad k \geq M.$$

From this, we obtain

$$\begin{aligned}F(\Omega_1, ..., \Omega_n) &\leqq \overline{\lim_{k\to\infty}}\, F(V_1^k(\lambda), ..., V_n^k(\lambda)) \\ &\leqq \lambda F(A_1, ..., A_n) + (1-\lambda)\, F(\Omega_1, ..., \Omega_n)\end{aligned}$$

for all $0 < \lambda < r$. This implies

$$F(\Omega_1, ..., \Omega_n) \leqq F(A_1, ..., A_n).$$

Since $(A_1, ..., A_n) \in \mathscr{S}$ is arbitrary, this shows that $(\Omega_1, ..., \Omega_n)$ is a global minimum of F on $\mathscr{S}$. Q.E.D.

In order to obtain the main result, we need the following Farkas–Minkowski type theorem for n-set functions.

THEOREM 2. *Let $\mathscr{S}$ be a convex subfamily of Γ^n,*

$$F = (F_1, ..., F_m) \colon \mathscr{S} \to \mathbb{R}^m \text{ be } \mathbb{R}_+^m\text{-convex}$$

and

$$G = (G_1, ..., G_p) \colon \mathscr{S} \to \mathbb{R}^p \text{ be } \mathbb{R}_+^p\text{-convex}.$$

If the system

$$\begin{cases} F(\Omega_1, ..., \Omega_n) < 0 \\ G(\Omega_1, ..., \Omega_n) < 0 \end{cases}$$

has no solution in $\mathscr{S}$, *then there exists* $u = (u_1, ..., u_m) \in \mathbb{R}^m_+$, $v = (v_1, ..., v_p) \in \mathbb{R}^p_+$, $(u, v) \neq (0, 0)$ *such that*

$$\sum_{i=1}^{m} u_i F_i(\Lambda_1, ..., \Lambda_n) + \sum_{i=1}^{p} v_i G_i(\Lambda_1, ..., \Lambda_n) \geqq 0$$

for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$.

Proof. Let $A = \{(y, z) \in \mathbb{R}^m \times \mathbb{R}^p \mid \text{there exists } (\Omega_1, ..., \Omega_n) \in \mathscr{S} \text{ such that } F(\Omega_1, ..., \Omega_n) < y \text{ and } G(\Omega_1, ..., \Omega_n) < z\}$. It is obvious that A does not contain the origin of $\mathbb{R}^m \times \mathbb{R}^p$. To show that A is convex in $\mathbb{R}^m \times \mathbb{R}^p$, let (y, z) and $(\bar{y}, \bar{z})$ be in A, then there exist $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$ and $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ such that

$$F(\Omega_1, ..., \Omega_n) < y, \qquad G(\Omega_1, ..., \Omega_n) < z$$

and

$$F(\Lambda_1, ..., \Lambda_n) < \bar{y}, \qquad G(\Lambda_1, ..., \Lambda_n) < \bar{z}.$$

It follows from the convexity of F and G on the convex subfamily $\mathscr{S}$ of Γ^n, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathscr{S}$ for all $k \in N$, and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1 - \lambda) F(\Omega_1, ..., \Omega_n)$$
$$< \lambda \bar{y} + (1 - \lambda) y$$

and

$$\overline{\lim_{k \to \infty}} G(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda G(\Lambda_1, ..., \Lambda_n) + (1 - \lambda) G(\Omega_1, ..., \Omega_n)$$
$$< \lambda \bar{z} + (1 - \lambda) z.$$

Therefore, there exists an integer $M > 0$ such that

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < \lambda \bar{y} + (1 - \lambda) y$$

and

$$G(V_1^k(\lambda), ..., V_n^k(\lambda)) < \lambda \bar{z} + (1 - \lambda) z$$

for $k \geqslant M$. Hence

$$\lambda(\bar{y}, \bar{z}) + (1-\lambda)(y, z) = (\lambda\bar{y} + (1-\lambda)\, y, \lambda\bar{z} + (1-\lambda)z) \in A.$$

It is obvious that A has a nonempty interior. Since $(0, 0) \notin A$, it follows from the separation theorem that there exist $u = (u_1, ..., u_m) \in \mathbb{R}^m$, $v = (v_1, ..., v_p) \in \mathbb{R}^m$ such that $(u, v) \neq (0, 0)$ and

$$\sum_{i=1}^{m} u_i y_i + \sum_{i=1}^{p} v_i z_i \geq 0 \qquad \text{for all} \quad (y, z) \in A, \tag{3}$$

where $Y = (y_1, ..., y_m)$, $z = (z_1, ..., z_p)$.

Following a similar argument as in Lemma 3.1 [1] we can show that $u \geqq 0$, $v \geqq 0$, and

$$\sum_{i=1}^{m} u_i F_i(A_1, ..., A_n) + \sum_{i=1}^{p} v_i G_i(A_1, ..., A_n) \geqq 0$$

for all $(A_1, ..., A_n) \in \mathscr{S}$. Q.E.D.

DEFINITION 3.2. Let $\mathscr{S}$ be a nonempty subfamily of Γ^n and $F: \mathscr{S} \to \mathbb{R}^m$. Then $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$ is called a weak local minimum of F on $\mathscr{S}$ if there exists $\delta > 0$ such that there does not exist $(A_1, ..., A_n) \in \mathscr{S}$ with $d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) < \delta$ and $F(A_1, ..., A_n) < F(\Omega_1, ..., \Omega_n)$. $(\Omega_1, ..., \Omega_n)$ is called a weak minimum of F on $\mathscr{S}$ if there does not exist $(A_1, ..., A_n) \in \mathscr{S}$ such that $F(A_1, ..., A_n) < F(\Omega_1, ..., \Omega_n)$.

Remark. It follows from Definitions 3.1 and 3.2 that if $F: \mathscr{S} \to \mathbb{R}$ and $(\Omega_1, ..., \Omega_n)$ is a weak local minimum of F on $\mathscr{S}$, then it is a local minimum of F on $\mathscr{S}$.

Applying Theorem 2, we have the following theorem.

THEOREM 3. *Let $\mathscr{S}$ be a convex subfamily of Γ^n and $F = (F_1, ..., F_m)$: $\mathscr{S} \to \mathbb{R}^m$, $G = (G_1, ..., G_p)$; $\mathscr{S} \to \mathbb{R}^p$ are differentiable at $(\Omega_1, ..., \Omega_n)$. Assume that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to problem* (P). *Then there exists nonzero element*

$$(\lambda, u) = ((\lambda_1, ..., \lambda_m), (u_1, ..., u_p)) \in \mathbb{R}^m_+ \times \mathbb{R}^p_+$$

such that

$$\sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n) = 0$$

and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j < f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} > + \sum_{j=1}^{m} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geq 0$$

for all $(A_1, ..., A_n) \in \mathcal{S}$.

Proof. Define

$$H_1(A_1, ..., A_n) = \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right)$$

$$H_2(A_1, ..., A_n) = \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) + G(\Omega_1, ..., \Omega_n).$$

It is obvious that H_1 is $\mathbb{R}_+^m$-convex and H_2 is $\mathbb{R}_+^p$-convex.

We claim that the system

$$\begin{cases} H_1(A_1, ..., A_n) < 0 \\ H_2(A_1, ..., A_n) < 0 \end{cases} \tag{4}$$

has no solution. If $(A_1, ..., A_n) \in \mathcal{S}$ were a solution of (4), fix $\lambda \in [0, 1]$; since $\mathcal{S}$ is a convex subfamily of Γ^n, it follows that there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with (Ω_i, A_i, λ) for $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \mathcal{S}$ for all $k \in N$. Then by the differentiability of F and G at $(\Omega_1, ..., \Omega_n)$, we would have

$$\begin{aligned} F(V_1^k(\lambda), ..., V_n^k(\lambda)) = F(\Omega_1, ..., \Omega_n) &+ \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle \right) \\ &+ E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \end{aligned} \tag{5}$$

and

$$\begin{aligned} G(V_1^k(\lambda), ..., V_n^k(\lambda)) = G(\Omega_1, ..., \Omega_n) &+ \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle \right) \\ &+ \tilde{E}((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)), \end{aligned} \tag{6}$$

where $E((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$ and $\tilde{E}((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$ are $o(d[(V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)])$. If we express

$$E((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) = (E_1((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)), \ldots, E_m((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))).$$

Then $E_i((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$ is $o(d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)))$ for each $i = 1, \ldots, m$. Therefore for each $\varepsilon > 0$ and $i = 1, \ldots, m$, there exists $r > 0$ such that $|E_i((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))| \leq \varepsilon d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$ for $d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) < r$. Let $\delta = r/d((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n))$. Then $\lim_{k \to \infty} d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) = \lambda d((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n))$ implies that for $\lambda < \delta$ and for sufficiently large k, we have $d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) < r$. Hence

$$|E_i((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))| \leq \varepsilon d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$$

for each $i = 1, \ldots, m$. This shows that $\overline{\lim}_{k \to \infty} E_i((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))$ is $o(\lambda)$ for each $i = 1, \ldots, m$ and therefore

$$\begin{aligned} &\overline{\lim_{k \to \infty}}\; E((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) \\ &\quad = (\overline{\lim_{k \to \infty}}\; E_1((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)), \ldots, \\ &\qquad \overline{\lim_{k \to \infty}}\; E_m((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n))) \\ &\quad = o(\lambda) \end{aligned} \tag{7}$$

Similarly $\overline{\lim}_{k \to \infty} \tilde{E}((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) = o(\lambda)$. It follows from (5), (6), and (7) that

$$\begin{aligned} &\overline{\lim_{k \to \infty}}\; F(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \\ &\quad = F(\Omega_1, \ldots, \Omega_n) + \lambda \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + o(\lambda) \\ &\quad = F(\Omega_1, \ldots, \Omega_n) + \lambda H_1(\Lambda_1, \ldots, \Lambda_n) + o(\lambda) \end{aligned}$$

and

$$\begin{aligned} &\overline{\lim_{k \to \infty}}\; G(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \\ &\quad = G(\Omega_1, \ldots, \Omega_n) + \lambda \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + o(\lambda) \\ &\quad = (1 - \lambda)\, G(\Omega_1, \ldots, \Omega_n) + \lambda H_2(\Lambda_1, \ldots, \Lambda_n) + o(\lambda). \end{aligned}$$

Since $H_1(\Lambda_1, ..., \Lambda_n) < 0$ and $H_2(\Lambda_1, ..., \Lambda_n) < 0$, for any $\delta > 0$, we can choose a small $\lambda' > 0$ and a natural number k such that

$$F(V_1^k(\lambda'), ..., F_n^k(\lambda')) < F(\Omega_1, ..., \Omega_n)$$
$$G(V_1^k(\lambda'), ..., V_n^k(\lambda')) < (1 - \lambda')\, G(\Omega_1, ..., \Omega_n) \leqq 0$$

and

$$d((V_1^k(\lambda'), ..., V_n^k(\lambda')), (\Omega_1, ..., \Omega_n)) < \delta.$$

This contradicts the assumption that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to (P). Hence system (4) does not have a solution. It follows from Theorem 2 that there exists a nonzero element $(\lambda, u) = ((\lambda_1, ..., \lambda_m), (u_1, ..., u_p)) \in \mathbb{R}_+^m \times \mathbb{R}_+^p$ such that

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$+ \sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n) \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}. \tag{8}$$

Letting $(\Lambda_1, ..., \Lambda_n) = (\Omega_1, ..., \Omega_n)$ in (8), we obtain

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \geq 0.$$

Since $u \geqq 0$ and $G(\Omega_1, ..., \Omega_n) \leqq 0$, it must be

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \leq 0.$$

It then reduces to

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) = 0.$$

Then by (8), we get

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$+ \sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) \geq 0$$

for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$. Q.E.D.

Remark. Since weak minimum is different from Pareto minimum, our result is different from Theorem 1 [13]. For $m=1$, Theorem 3 reduces to Theorem 3.7 [4].

If we give an additional condition of regularity for the inequality constraint, then we get

THEOREM 4. *In Theorem* 3, *if we assume further that there exists a* $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \mathscr{S}$ *such that*

$$G(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i} - \Omega_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle \right) < 0,$$

then there exists $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ *such that*

$$w[G(\Omega_1, ..., \Omega_n)] = 0$$

and

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) + w\left[\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right] < 0$$

fail to hold for any $(A_1, ..., A_n) \in \mathscr{S}$.

Proof. It follows from Theorem 3, that there exists nonzero $(\lambda, u) = ((\lambda_1, ..., \lambda_m), (u_1, ..., u_p)) \in \mathbb{R}_+^m \times \mathbb{R}_+^p$ such that

$$\sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n) = 0$$

and

$$\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{m} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geqq 0, \qquad (9)$$

for all $(A_1, ..., A_n) \in \mathscr{S}$.

By assumption, there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \mathscr{S}$ such that

$$G(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle \right) < 0.$$

If $\lambda = 0$, then $u \neq 0$ and $u \geqq 0$ and so $\sum_{i=1}^{p} u_i z_i > 0$ for all $z = (z_1, ..., z_p) \in \mathbb{R}^p$ and $z > 0$. Thus, by assumption, $\lambda = 0$, we should get

$$0 > \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} u_j G_j(\Omega_1, ..., \Omega_n)$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j (g_*^{ij}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i}) \geq 0.$$

This is a contradiction; therefore $\lambda \neq 0$. Since $\lambda \geqq 0$ and $\lambda \neq 0$, we can choose $v = (v_1, ..., v_m) \in \mathbb{R}^m$ and $v > 0$ such that

$$\sum_{i=1}^{m} \lambda_i v_i = 1.$$

Define $w = (w_1, ..., w_m): \mathbb{R}^p \to \mathbb{R}^m$ by

$$w(z) = \left(\sum_{i=1}^{p} u_i z_i \right) v,$$

where $z = (z_1, ..., z_p) \in \mathbb{R}^p$. Then $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ and $w[G(\Omega_1, ..., \Omega_n)] = [\sum_{i=1}^{p} u_i G_i(\Omega_1, ..., \Omega_n)] v = 0$. By (9), we obtain

$$\sum_{j=1}^{m} \lambda_j \left[\sum_{i=1}^{n} \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right]$$

$$= \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{p} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{A_i} - \chi_{\Omega_i} \rangle \geq 0.$$

Since $\lambda \geqq 0$ and $\lambda \neq 0$, this shows that

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) + w \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not holds for any $(A_1, ..., A_n) \in \mathcal{S}$. Q.E.D.

DEFINITION 3.3. A differentiable set function $F=(F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ is said to be locally convex at $(\Omega_1, ..., \Omega_n)$ if there exists $\delta > 0$ such that

$$F(\Lambda_1, ..., \Lambda_n) \geqq F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right)$$

for all $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$ with $d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta$.

The following theorem gives a sufficient conditions for the existence of a weak local minimum to problem (P).

THEOREM 5. *Suppose that the set function* $F=(F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ *and* $G=(G_1, ..., G_p): \mathscr{S} \to \mathbb{R}^p$ *are differentiable and locally convex at* $(\Omega_1, ..., \Omega_n)$. *If there exists* $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ *such that* $w(G(\Omega_1, ..., \Omega_n)) = 0$ *and*

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + w\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) < 0$$

does not hold for any $(\Lambda_1, ..., \Lambda_n) \in \mathscr{S}$, *then* $(\Omega_1, ..., \Omega_n)$ *is a weak local minimum to* (P).

Proof. Let $w=(w_1, ..., w_m) \in B^+(\mathbb{R}^p, \mathbb{R}^m)$, then

$$w_i \in B^+(\mathbb{R}^p, \mathbb{R}^1) \qquad \text{for each} \quad i = 1, ..., m.$$

Let

$$H_j(\Lambda_1, ..., \Lambda_n) = \sum_{i=1}^{n} \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + w_j\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right).$$

It is easy to see that $H_j: \mathscr{S} \to \mathbb{R}^1$ is convex and the system

$$\begin{cases} H_1(\Lambda_1, ..., \Lambda_n) < 0 \\ \vdots \\ H_m(\Lambda_1, ..., \Lambda_n) < 0 \end{cases}$$

does not have a solution, then it follows from Theorem 2 that there exists nonzero

$$\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}_+^m$$

such that

$$\sum_{i=1}^{m} \lambda_i H_i(\Lambda_1, ..., \Lambda_n) \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}.$$

That is,

$$\begin{aligned} &\sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \\ &\qquad + \sum_{j=1}^{m} \lambda_j w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\ &\qquad \geqq 0, \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S}. \end{aligned} \tag{10}$$

Since F and G are locally convex at $(\Omega_1, ..., \Omega_n)$, there exists $\delta > 0$ such that

$$\begin{aligned} F(\Lambda_1, ..., \Lambda_n) \geqq F(\Omega_1, ..., \Omega_n) \\ + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \end{aligned} \tag{11}$$

and

$$\begin{aligned} G(\Lambda_1, ..., \Lambda_n) \geqq G(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\ \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S} \\ \text{with} \quad d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta. \end{aligned} \tag{12}$$

By (10), (11), (12), and $w(G(\Omega_1, ..., \Omega_n)) = 0 = (w_1(G(\Omega_1, ..., \Omega_n)), ..., w_m(G(\Omega_1, ..., \Omega_n)))$, we have

$$\begin{aligned} &\sum_{j=1}^{m} \lambda_j [F_j(\Lambda_1, ..., \Lambda_n) - F_j(\Omega_1, ..., \Omega_n)] \\ &\qquad \geq \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_j \langle f_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \\ &\qquad \geq - \sum_{j=1}^{m} \lambda_j w_j \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \\ &\qquad \geq - \sum_{j=1}^{m} \lambda_j w_j (G(\Lambda_1, ..., \Lambda_n)) + \sum_{j=1}^{m} \lambda_j w_j (G(\Omega_1, ..., \Omega_n)) \\ &\qquad = - \sum_{j=1}^{m} \lambda_j w_j (G(\Lambda_1, ..., \Lambda_n)) \\ &\qquad \geq 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \mathscr{S} \\ &\qquad \qquad \text{with} \quad d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta \end{aligned} \tag{13}$$

Since $\lambda \geqq 0$, $\lambda \neq 0$, it follows from (13) that there exists no $(A_1, ..., A_n) \in \mathscr{S}$ with $G(A_1, ..., A_n) \leqq 0$ and $d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) < \delta$ such that

$$F(A_1, ..., A_n) < F(\Omega_1, ..., \Omega_n).$$

This shows that $(\Omega_1, ..., \Omega_n)$ is a weak local minimum to (P). Q.E.D.

The following corollary follows immediately from Theorems 3 and 5.

COROLLARY 6. *Let $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ be differentiable and locally convex at $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$, then $(\Omega_1, ..., \Omega_n)$ is a weak local minimum of F on $\mathscr{S}$ if and only if*

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i}\rangle\right) < 0$$

does not hold for any $(A_1, ..., A_n) \in \mathscr{S}$.

Following a similar argument as in the proof of Theorem 4.5 [4], we have

LEMMA 7. *Let $F = (F_1, ..., F_m): \mathscr{S} \to \mathbb{R}^m$ be differentiable and convex on $\mathscr{S}$, then for all $(A_1, ..., A_n), (\Omega_1, ..., \Omega_n) \in \mathscr{S}$,*

$$F(A_1, ..., A_n)) \geq F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i}\rangle\right).$$

Remark. It follows from Lemma 7 that if $F: \mathscr{S} \to \mathbb{R}^m$ is differentiable and convex on $\mathscr{S}$, then F is locally convex at any $(\Omega_1, ..., \Omega_n) \in \mathscr{S}$.

Applying Lemma 7 and following similar arguments as in the proof of Theorem 6, we have

THEOREM 8. *Suppose that the set function $F: \mathscr{S} \to \mathbb{R}^m$ and $G: \mathscr{S} \to \mathbb{R}^p$ are convex and differentiable on $\mathscr{S}$. If there exists $w \in B^+(\mathbb{R}^p, \mathbb{R}^m)$ such that*

$$w(G(\Omega_1, ..., \Omega_n)) = 0$$

and

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{A_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle f_*^{im}, \chi_{A_i} - \chi_{\Omega_i}\rangle\right) + w\left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{A_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle g_*^{ip}, \chi_{A_i} - \chi_{\Omega_i}\rangle\right) < 0$$

does not hold for any $(A_1, ..., A_n) \in \mathscr{S}$, then $(\Omega_1, ..., \Omega_n)$ is a weak minimum to (P).

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