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DUALITY THEOREMS OF VECTOR-VALUED n-SET FUNCTIONS

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Abstract—The Wolfe and Mond-Weir types of dual problems for Convex vector-valued n-set functions are formulated and we prove the weak duality, duality and converse duality theorems for these types of problems.

1. INTRODUCTION

Throughout this paper, we assume (X, Γ, μ) is a finite atomless measure space, with $L_1(X, \Gamma, \mu)$ separable. We denote $\Gamma^n = \{(\Lambda_1, \dots, \Lambda_n); \Lambda_i \in \Gamma, 1 \le i \le n\}$ and $S_1 \times \dots \times S_n = \{(\Lambda_1, \dots, \Lambda_n); \Lambda_i \in S_i, 1 \le i \le n\} \subseteq \Gamma^n$. We consider a multiobjective optimization problem as follows:

$$\begin{array}{ll} \text{Minimize} & F(\Lambda_1, \dots, \Lambda_n) \\ \text{subject to} & (\Lambda_1, \dots, \Lambda_n) \in S_1 \times \dots \times S_n \quad \text{and} \quad G(\Lambda_1, \dots \Lambda_n) \le 0, \end{array} \tag{P1}$$

where $F : S_1 \times \dots \times S_n \to \mathbf{R}^p$, $G : S_1 \times \dots \times S_n \to \mathbf{R}^m$ are n-set functions. In [1], we prove the duality theorem when the real valued objective function is a set function of a single set. In this paper, the Wolfe and Mond-Weir types of dual problems are formulated. We prove the duality, weak duality and converse duality theorems when the objective and constraint functions are vector-valued n-set functions. We also consider the case of convex n-set functions and the special case when the set functions are differentiable n-set functions.

2. PRELIMINARIES

Throughout the paper, we assume that (X, Γ, μ) is a finite atomless measured space with $L_1(X, \Gamma, \mu)$ separable. For two vectors $x = (x_1, \dots, x_p)$ and $y = (y_1, \dots, y_p)$ in a p-dimensional Euclidean space $\mathbf{R}^p$, we denote $\langle x, y\rangle = \sum_{i=1}^{p} x_i\, y_i$, and

(1) $x < y$ iff $x_i < y_i$, for all $i = 1, \dots, p$.
(2) $x \le y$ iff $x_i \le y_i$, for all $i = 1, \dots, p$ and $x \ne y$.
(3) $x \leqq y$ iff $x_i \leqq y_i$, for all $i = 1, \dots, p$.

We also denote $0 = (0, \dots, 0)$ in $\mathbf{R}^p$, $\mathbf{R}^p_+ = \{x \in \mathbf{R}^p; x \geqq 0\}$ and $\mathbf{R}^p_- = \{x \in \mathbf{R}^p; x \leqq 0\}$. We define a pseudometric d on $\Gamma^n = \Gamma \times \dots \times \Gamma = \{(\Lambda_1, \dots, \Lambda_n) \mid \Lambda_i \in \Gamma, 1 \le i \le n\}$ in the following way, $d[(\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n)] = \{ \sum_{i=1}^{n} [\mu(\Omega_i \,\Delta\, \Lambda_i)]^2 \}^{1/2}$, where $(\Omega_1, \dots, \Omega_n)$ and $(\Lambda_1, \dots, \Lambda_n) \in \Gamma^n$ and $\Omega_i \,\Delta\, \Lambda_i$ denote the symmetric difference for Ω_i and Λ_i. For $f \in L_1(X, \Gamma, \mu)$, the integral $\int_\Omega f\, d\mu$ will be denoted by $\langle f, \chi_\Omega\rangle$, where χ_Ω denotes the characteristic function of Ω. For a set $E \subseteq \mathbf{R}^p$, int E and $\bar{E}$ will denote the interior points and closure of E in $\mathbf{R}^p$, respectively.

DEFINITION 2.1. *A point $x^* \in E \subset \mathbf{R}^p$ is a lower (resp. upper) efficient point if there is no $x \le x^*$ (resp. $x \ge x^*$). We denote the set of all lower efficient points (resp. upper efficient points) of E by $\underline{e}(E)$ (resp. $\bar{e}(E)$).*

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LEMMA 2.2 [2]. *Suppose that for a point $x^* \in E \subseteq \mathbf{R}^p$, there exists a $\hat{\mu} \in int\, \mathbf{R}^p_+$ such that $\langle \hat{\mu}, x^* \rangle \geq \langle \hat{\mu}, x \rangle$ (resp. $\langle \hat{\mu}, x^* \rangle \leq \langle \hat{\mu}, x^* \rangle$) for all $x \in E$, then $x^* \in \bar{e}(E)$ (resp. $x^* \in \underline{e}(E)$).*

DEFINITION 2.3. *A set function $F : \Gamma \to \mathbf{R}$ is differentiable at $\Omega \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$, the derivative of f such that*

$$F(\Lambda) = F(\Omega) + \langle f, \chi_\Lambda - \chi_\Omega \rangle + \mu(\Omega \,\Delta\, \Lambda)\, E(\Omega, \Lambda), \quad \text{for all } \Lambda \in \Gamma,$$
$$\text{where} \lim_{\mu(\Omega \Delta \Lambda) \to 0} E(\Omega, \Lambda) = 0.$$

DEFINITION 2.4. *Let $F : \Gamma^n \to \mathbf{R}$ and $(\Omega_1, \dots, \Omega_n) \in \Gamma^n$. Then F is said to have partial derivative with respect to Λ_i if the set function*

$$H(\Lambda_i) = F(\Omega_1, \dots, \Omega_{i-1}, \Lambda_i, \Omega_{i+1}, \dots, \Omega_n)$$

has derivative h_{Ω_i} at Ω_i. In this case, we define the i-th partial derivative of F at $(\Omega_1, \dots, \Omega_n)$ to be $f^i_{\Omega_1,\dots,\Omega_n} = h_{\Omega_i}$.

We define the derivative of vector valued n-set functions as follows:

DEFINITION 2.5. *Let $S_1 \times \cdots \times S_n \subset \Gamma^n$, $F = (F_1, \dots, F_p)$: $S_1 \times \cdots \times S_n \to \mathbf{R}^p$ and $(\Omega_1, \dots, \Omega_n) \in S_1 \times \cdots \times S_n$. Then F is said to be differentiable at $(\Omega_1, \dots, \Omega_n)$ if the partials $f^{ij}_{\Omega_1,\dots,\Omega_n}$, $i = 1, \dots, n$, of F_j exist for each $j = 1, \dots, p$ and satisfy*

$$F(\Lambda_1, \dots, \Lambda_n) = F(\Omega_1, \dots, \Omega_n) + \left(\sum_{i=1}^{n} \langle f^{i1}_{\Omega_1,\dots,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \dots, \sum_{i=1}^{n} \langle f^{ip}_{\Omega_1,\dots,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) + W_F((\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n)),$$

for all $(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$, where

$$\frac{W_F((\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n))}{d((\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n))} \to 0,$$

as $d((\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n)) \to 0$. If F is differentiable at every point $(\Omega_1, \dots, \Omega_n)$ of $S_1 \times \cdots \times S_n$, we say F is differentiable on $S_1 \times \cdots \times S_n$.

As for the definition and properties of Morris sequence, please refer to [3,4].

DEFINITION 2.6. *A subfamily $S_1 \times \cdots \times S_n$ of Γ^n is convex if, given $(\Omega_1, \dots, \Omega_n), (\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$ and $\lambda \in [0,1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in S_i associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, \dots, n$, such that $(V_1^k(\lambda), \dots, V_n^k(\lambda)) \in S_1 \times \cdots \times S_n$ for all $k \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers.*

DEFINITION 2.7. *A set function $F = (F_1, \dots, F_p) : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ is called convex on a convex subfamily $S_1 \times \cdots \times S_n$ of Γ^n if for each $(\Omega_1, \dots, \Omega_n)$ and $(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$, $\lambda \in [0,1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, \dots, n$ such that $(V_1^k(\lambda), \dots, V_n^k(\lambda)) \in S_1 \times \cdots \times S_n$ for all $k \in \mathbf{N}$ and*

$$\overline{\lim_{k\to\infty}} F(V_1^k(\lambda), \dots, V_n^k(\lambda)) = (\overline{\lim_{k\to\infty}} F(V_1^k(\lambda), \dots, V_n^k(\lambda)), \dots, \overline{\lim_{k\to\infty}} F(V_1^k(\lambda), \dots, V_n^k(\lambda)))$$
$$\leq \lambda F(\Lambda_1, \dots, \Lambda_n) + (1 - \lambda) F(\Omega_1, \dots, \Omega_n).$$

DEFINITION 2.8. *$(\Omega_1, \dots, \Omega_n) \in A$ is said to be a proper $\mathbf{R}^p_+$-solution of (P1) if $\overline{F(A) + \mathbf{R}^p_+ - F(\Omega_1, \dots, \Omega_n)} \cap \mathbf{R}^p_- = \{0\}$, where $A = \{(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n,\ G(\Lambda_1, \dots, \Lambda_n) \leq 0\}$.*

LEMMA 2.9. [5, Lemma 2.4]. *Let E be a $\mathbf{R}^p_+$-convex set, then $y_0 \in E$ satisfies*

$$\overline{E + \mathbf{R}^p_+ - y_0} \cap \mathbf{R}^p_- = \{0\},$$

iff there exists a vector $\mu \in \text{int } \mathbf{R}_+^p$ *such that*

$$\langle \mu, y_0 \rangle \le \langle \mu, y \rangle, \qquad \text{for any } y \in E.$$

LEMMA 2.10 [6]. *Let* $F = (F_1, \dots, F_p) : \Gamma^n \to \mathbf{R}^p$ *be convex and differentiable on* Γ^n, *then for all* $(\Lambda_1, \dots, \Lambda_n)$, $(\Omega_1, \dots, \Omega_n) \in \Gamma^n$,

$$F(\Lambda_1, \dots, \Lambda_n) \ge F(\Omega_1, \dots, \Omega_n) + \left(\sum_{i=1}^n \langle f^{i1}_{\Omega_1,\dots,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \dots, \sum_{i=1}^n \langle f^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right),$$

where $f^{ij}_{\Omega_1,\dots,\Omega_n}$ *denotes the i-th partial derivative of* F_j *at* $(\Omega_1, \dots, \Omega_n)$.

DEFINITION 2.11. *A set function* $F : \Gamma^n \to \mathbf{R}^p$ *is said to be* w^**-continuous on dom* F *if for any* $(\Omega_1, \dots, \Omega_n) \in$ *dom* F *and for each* $i = 1, \dots, n$, $\chi_{\Omega_i^m} \xrightarrow{w^*} \chi_{\Omega_i}$ *as* $m \to \infty$, *implies* $\lim_{m\to\infty} F(\Omega_1^m, \dots, \Omega_n^m) = F(\Omega_1, \dots, \Omega_n)$.

REMARK 2.12. If F is differentiable at $(\Omega_1, \dots, \Omega_n) \in \Gamma^n$, then it follows from the definitions of derivative and w^*-continuity that F is w^*-continuous at $(\Omega_1, \dots, \Omega_n)$. Throughout this paper, we denote the set A by

$$A = \{(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n,\ G(\Lambda_1, \dots, \Lambda_n) \le 0\}.$$

DEFINITION 2.13. $(\Omega_1, \dots, \Omega_n) \in A$ *is called a Geoffrion properly efficient solution of problem (P1) if* $(\Omega_1, \dots, \Omega_n) \in \underline{e}[F(A)]$ *and, if there exists* $M > 0$ *such that for each* i *and* $(\Lambda_1, \dots, \Lambda_n) \in A$ *satisfying* $F_i(\Lambda_1, \dots, \Lambda_n) < F_i(\Omega_1, \dots, \Omega_n)$, *there exists* j *with* $F_j(\Lambda_1, \dots, \Lambda_n) > F_j(\Omega_1, \dots, \Omega_n)$ *and*

$$\frac{F_i(\Omega_1, \dots, \Omega_n) - F_i(\Lambda_1, \dots, \Lambda_n)}{F_j(\Lambda_1, \dots, \Lambda_n) - F_j(\Omega_1, \dots, \Omega_n)} \le M.$$

LEMMA 2.14 [7]. *If* $\overline{F(A)}$ *is* $\mathbf{R}_+^p$*-convex, then* $(\Omega_1, \dots, \Omega_n)$ *is a properly* $\mathbf{R}_+^p$*-solution if and only if* $(\Omega_1, \dots, \Omega_n)$ *is a Geoffrion properly efficient solution.*

LEMMA 2.15 [8]. *Let* $S_1 \times \cdots \times S_n$ *be a convex subfamily of* Γ^n *and* $F : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ *be convex and* w^**-continuous,* $G : S_1 \times \cdots \times S_n \to \mathbf{R}^m$ *be convex. Suppose that there exists* $(\hat{\Omega}_1, \dots, \hat{\Omega}_n) \in S_1 \times \cdots \times S_n$ *such that* $G(\hat{\Omega}_1, \dots, \hat{\Omega}_n) < 0$, *then* $\overline{F(A)}$ *is* $\mathbf{R}_+^p$*-convex.*

DEFINITION 2.16. *The element* $(f_1, \dots, f_n) \in L_1(X, \Gamma, \mu) \times \cdots \times L_1(X, \Gamma, \mu)$ *is called a subgradient of a convex n-set functions* $F : S_1 \times \cdots \times S_n \to \mathbf{R}$ *at* $(\Omega_1, \dots, \Omega_n) \in \Gamma^n$, *if it satisfies the inequality*

$$F(\Lambda_1, \dots, \Lambda_n) \ge F(\Omega_1, \dots, \Omega_n) + \sum_{i=1}^n \langle f_i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle,$$

for all $(\Lambda_1, \dots, \Lambda_n) \in \Gamma^n$. *The set of all subgradients of* F *at* $(\Omega_1, \dots, \Omega_n)$ *is denoted by* $\partial F(\Omega_1, \dots, \Omega_n)$ *and is called the subdifferential of* F *at* $(\Omega_1, \dots, \Omega_n)$. *If* $\partial F(\Omega_1, \dots, \Omega_n) \ne \emptyset$, *then* F *is called subdifferentiable at* $(\Omega_1, \dots, \Omega_n)$.

REMARK 2.7. It follows from Lemma 2.10 that if $F : \Gamma^n \to \mathbf{R}$ is differentiable and convex on Γ^n then $(f_1, \dots, f_n) \in \partial F(\Omega_1, \dots, \Omega_n)$ and F is subdifferentiable at $(\Omega_1, \dots, \Omega_n)$.

3. MAIN RESULTS

In this paper, we present and prove the Wolfe and Mond-Weir duality of vector-valued n-set functions. The Wolfe duality problem of convex n-set functions can be formulated as follows:

$$\begin{aligned}
\text{Maximize} \quad & F(\Lambda_1, \dots, \Lambda_n) + \langle \beta, G(\Lambda_1, \dots, \Lambda_n) \rangle e \\
\text{subject to} \quad & \lambda = (\lambda_1, \dots, \lambda_p) \in \text{int } \mathbf{R}_+^p,\ \beta = (\beta_1, \dots, \beta_m) \in \mathbf{R}_+^m, && \text{(D1)} \\
& (\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n, \\
& \sum_{i=1}^n \left\langle \sum_{j=1}^p \lambda_j f^{ij} + \sum_{j=1}^m \beta_j g^{ij}, \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \ge 0, && (1)
\end{aligned}$$

for all $(B_1,\ldots,B_n) \in S_1 \times \cdots \times S_n$, and for some $(f^{1j},\ldots,f^{nj}) \in \partial F_j(\Lambda_1,\ldots,\Lambda_n)$, $(g^{1i},\ldots,g^{ni}) \in \partial G_i(\Lambda_1,\ldots,\Lambda_n)$, $i = 1,\ldots,m$, $j = 1,\ldots,p$, where $F : S_1 \times \cdots \times S_n \to \mathbf{R}^p$, $G : S_1 \times \ldots S_n \to \mathbf{R}^m$ are convex n-set functions and $e = (1,\ldots,1)$.

For the differentiable n-set function, we consider the problem (P2).

$$\begin{aligned} &\text{Minimize} && F(\Lambda_1,\ldots,\Lambda_n) \\ &\text{subject to} && (\Lambda_1,\ldots,\Lambda_n) \in \Gamma^n,\ G(\Lambda_1,\ldots,\Lambda_n) \le 0. \end{aligned} \tag{P2}$$

The Wolfe dual problem of (P2) is as follows:

$$\begin{aligned} &\text{Maximize} && F(\Lambda_1,\ldots,\Lambda_n) + \langle \beta, G(\Lambda_1,\ldots,\Lambda_n)\rangle e \\ &\text{subject to} && \lambda = (\lambda_1,\ldots,\lambda_p) \in \operatorname{int} \mathbf{R}^p_+,\ \beta = (\beta_1,\ldots,\beta_m) \in \mathbf{R}^m_+, \\ &&& (\Lambda_1,\ldots,\Lambda_n) \in \Gamma^n, \\ &&& \sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j\, f^{ij}_{\Lambda_1,\ldots,\Lambda_n} + \sum_{j=1}^{m} \beta_j\, g^{ij}_{\Lambda_1,\ldots,\Lambda_n},\ \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \ge 0, \end{aligned} \tag{D2}$$

for all $(\Lambda_1,\ldots,\Lambda_n) \in \Gamma^n$, where $F : \Gamma^n \to \mathbf{R}^m$ are differentiable n-set functions, $f^{ij}_{\Lambda_1,\ldots,\Lambda_n}$, $g^{ij}_{\Lambda_1,\ldots,\Lambda_n}$ are the i-th partial derivatives of F_j and G_j at $(\Lambda_1,\ldots,\Lambda_n)$, respectively.

The Mond-Weir dual problem of (P1) with convex n-set functions can be formulated as follows:

$$\begin{aligned} &\text{Maximize} && F(\Lambda_1,\ldots,\Lambda_n) \\ &\text{subject to} && \lambda = (\lambda_1,\ldots,\lambda_n) \in \operatorname{int} \mathbf{R}^p_+,\ \beta = (\beta_1,\ldots,\beta_m) \in \mathbf{R}^m_+, \\ &&& (\Lambda_1,\ldots,\Lambda_n) \in S_1 \times \cdots \times S_n, \end{aligned} \tag{D3}$$

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j\, f^{ij} + \sum_{j=1}^{m} \beta_j\, g^{ij},\ \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \ge 0, \tag{2}$$

for all $(B_1,\ldots,B_n) \in S_1 \times \cdots \times S_n$ and for some $(f^{1j},\ldots,f^{nj}) \in \partial F_j(\Lambda_1,\ldots,\Lambda_n)$, $(g^{1i},\ldots,g^{ni}) \in \partial G_i(\Lambda_1,\ldots,\Lambda_n)$, $j = 1,\ldots,p$, $i = 1,\ldots,m$, $\langle \beta, G(\Lambda_1,\ldots,\Lambda_n)\rangle \ge 0$, (3)
where F and G are convex on Γ^n.

For the special case of (D3), when F, G are differentiable on Γ^n, the Mond-Weir type dual problem of (P2) can be formulated as follows:

$$\begin{aligned} &\text{Maximize} && F(\Lambda_1,\ldots,\Lambda_n) \\ &\text{subject to} && \lambda = (\lambda_1,\ldots,\lambda_p) \in \operatorname{int} \mathbf{R}^p_+,\ \beta = (\beta_1,\ldots,\beta_m) \in \mathbf{R}^m_+, \\ &&& (\Lambda_1,\ldots,\Lambda_n) \in \Gamma^n, \end{aligned} \tag{D4}$$

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j\, f^{ij}_{\Lambda_1,\ldots,\Lambda_n} + \sum_{j=1}^{m} \beta_j\, g^{ij}_{\Lambda_1,\ldots,\Lambda_n},\ \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \ge 0, \tag{4}$$

$$\text{for all } (B_1,\ldots,B_n) \in \Gamma^n, \quad \langle \beta, G(\Lambda_1,\ldots,\Lambda_n)\rangle \ge 0. \tag{5}$$

We say $((\Lambda_1,\ldots,\Lambda_n),\lambda,\beta)$ is a feasible solution to (D1) if $\lambda \in \operatorname{int} \mathbf{R}^p_+$, $\beta \in \mathbf{R}^m_+$, $(\Lambda_1,\ldots,\Lambda_n) \in S_1 \times \cdots \times S_n$ and (1) holds. We say $((\Omega_1,\ldots,\Omega_n),\lambda^*,\beta^*)$ is a Pareto optimal solution to (D1) if $((\Omega_1,\ldots,\Omega_n),\lambda^*,\beta^*)$ is a feasible solution to (D1) and there is no feasible solution $((\Lambda_1,\ldots,\Lambda_n),\lambda,\beta)$ to (D1) such that

$$F(\Lambda_1,\ldots,\Lambda_n) + \langle \beta, G(\Lambda_1,\ldots,\Lambda_n)\rangle\, e \ge F(\Omega_1,\ldots,\Omega_n) + \langle \beta^*, G(\Omega_1,\ldots,\Omega_n)\rangle\, e.$$

The feasible solution and Pareto optimal solution to (D2)–(D4) can be defined similarly.

THEOREM 1. *(Weak duality theorem). Let $(\Omega_1,\ldots,\Omega_n)$ be feasible for (P1) and let $((\Lambda_1,\ldots,\Lambda_n),\lambda,\beta)$ be feasible for (D1), then*

$$\langle \lambda, F(\Omega_1,\ldots,\Omega_n)\rangle \ge \langle \lambda, F(\Lambda_1,\ldots,\Lambda_n) + \langle \beta, G(\Lambda_1,\ldots,\Lambda_n)\rangle\, e\rangle.$$

PROOF. Since $((\Lambda_1,\dots,\Lambda_n),\lambda,\beta)$ is feasible for (D1), it follows that $(\Lambda_1,\dots,\Lambda_n)\in S_1\times\cdots\times S_n$, $\lambda=(\lambda_1,\dots,\lambda_p)\in \operatorname{int}\mathbf{R}^p_+$, $\beta=(\beta_1,\dots,\beta_m)\in\mathbf{R}^m_+$ and for each $j=1,\dots,p$, $i=1,\dots,m$, there exist $(f^{1j},\dots,f^{nj})\in\partial F_j(\Lambda_1,\dots,\Lambda_n)$, $(g^{1i},\dots,g^{ni})\in\partial G_i(\Lambda_1,\dots,\Lambda_n)$ such that

$$\sum_{i=1}^{n}\left\langle\sum_{j=1}^{p}\lambda_j f^{ij}+\sum_{j=1}^{m}\beta_j g^{ij},\ \chi_{B_i}-\chi_{\Lambda_i}\right\rangle\geq 0,$$

for all $(B_1,\dots,B_n)\in S_1\times\cdots\times S_n$.

Without loss of generality, we may assume that $\sum_{j=1}^{p}\lambda_j=1$.

$$\begin{aligned}
&\langle\lambda,F(\Omega_1,\dots,\Omega_n)\rangle-\langle\lambda,F(\Lambda_1,\dots,\Lambda_n)+\langle\beta,G(\Lambda_1,\dots,\Lambda_n)\rangle e\rangle\\
&\quad=\sum_{j=1}^{p}\lambda_j[F_j(\Omega_1,\dots,\Omega_n)-F_j(\Lambda_1,\dots,\Lambda_n)]\\
&\qquad+\langle\beta,G(\Omega_1,\dots,\Omega_n)-G(\Lambda_1,\dots,\Lambda_n)\rangle-\langle\beta,G(\Omega_1,\dots,\Omega_n)\rangle\\
&\quad=\sum_{j=1}^{p}\lambda_j\,[F_j(\Omega_1,\dots,\Omega_n)-F_j(\Lambda_1,\dots,\Lambda_n)]\\
&\qquad+\sum_{j=1}^{m}\beta_j\,[G_j(\Omega_j,\dots,\Omega_n)-G_j(\Lambda_1,\dots,\Lambda_n)]-\langle\beta,G(\Omega_1,\dots,\Omega_n)\rangle\\
&\quad\geq\sum_{j=1}^{p}\lambda_j\sum_{i=1}^{n}\langle f^{ij},\chi_{\Omega_i}-\chi_{\Lambda_i}\rangle+\sum_{j=1}^{m}\beta_j\sum_{i=1}^{n}\langle g^{ij},\chi_{\Omega_i}-\chi_{\Lambda_i}\rangle-\langle\beta,G(\Omega_1,\dots,\Omega_n)\rangle\\
&\quad=\sum_{i=1}^{n}\left\langle\sum_{j=1}^{p}\lambda_j f^{ij}+\sum_{j=1}^{m}\beta_j g^{ij},\ \chi_{\Omega_i}-\chi_{\Lambda_i}\right\rangle-\langle\beta,G(\Omega_1,\dots,\Omega_n)\rangle\geq 0,
\end{aligned}$$

for any feasible solution $(\Omega_1,\dots,\Omega_n)$ of (P1) and any feasible solution $((\Lambda_1,\dots,\Lambda_n),\lambda,\beta)$ of (D1). This completes the proof of the theorem. ∎

REMARK 2. It follows from Lemma 2.12 that if $F=(F_1,\dots,F_p):\Gamma^n\to\mathbf{R}^p$ are differentiable and convex, then for each $j=1,\dots,p$, $(f^{1j}_*,\dots,f^{nj}_*)\in\partial F_j(\Omega_1,\dots,\Omega_n)$, where f^{ij}_* denotes the i-th partial derivative of F_j at $(\Omega_1,\dots,\Omega_n)$.

COROLLARY 3. *(Weak duality). Let $F:\Gamma^n\to\mathbf{R}^p$, $G:\Gamma^n\to\mathbf{R}^m$ be convex and differentiable, $(\Omega_1,\dots,\Omega_n)$ be feasible for (P2) and $((\Lambda_1,\dots,\Lambda_n),\lambda,\beta)$ be feasible for (D2), then*

$$\langle\lambda,F(\Omega_1,\dots,\Omega_n)\rangle\geq\langle\lambda,F(\Lambda_1,\dots,\Lambda_n)+\langle\beta,G(\Lambda_1,\dots,\Lambda_n)\rangle e\rangle.$$

PROOF. If we let $S_1\times\cdots\times S_n=\Gamma^n$, then we see from Remark 2 that for each $i=1,\dots,m$, $j=1,\dots,p$, $(f^{nj}_{\Lambda_1,\dots,\Lambda_n},\dots,f^{nj}_{\Lambda_1,\dots,\Lambda_n})\in\partial F_j(\Lambda_1,\dots,\Lambda_n)$, $(g^{1i}_{\Lambda_1,\dots,\Lambda_n},\dots,g^{ni}_{\Lambda_1,\dots,\Lambda_n})\in\partial G_i\,(\Lambda_1,\dots,\Lambda_n)$.

Since $((\Lambda_1,\dots,\Lambda_n),\lambda,\beta)$ is feasible for (D2), it follows that $\lambda\in\operatorname{int}\mathbf{R}^p_+$, $\beta\in\mathbf{R}^m_+$, and there exist $(f^{1j}_{\Lambda_1,\dots,\Lambda_n},\dots,f^{nj}_{\Lambda_1,\dots,\Lambda_n})\in\partial F_j(\Lambda_1,\dots,\Lambda_n)$, $(g^{1j}_{\Lambda_1,\dots,\Lambda_n},\dots,g^{nj}_{\Lambda_1,\dots,\Lambda_n})\in\partial G_j(\Lambda_1,\dots,\Lambda_n)$ such that (1) holds and the corollary follows immediately from Theorem 1. ∎

LEMMA 4 [9]. *Let $F(F_1,\dots,F_p):S_1\times\cdots\times S_n\to\mathbf{R}^p$, $G=(G_1,\dots,G_m):S_1\times\cdots\times S_n\to\mathbf{R}^m$ be proper convex set functions on a convex subfamily $S_1\times\cdots\times S_n$ of Γ^n. Suppose that $F_1,\dots,F_p$ are w^*-continuous and $G_1,\dots,G_m$, except possibly one, are w^*-continuous on $S_1\times\cdots\times S_n$ and for each $i=1,\dots,n$, $\overline{S_i}$ contains a relative interior point. Suppose further that there exists $(\hat\Omega_1,\dots,\hat\Omega_n)\in S_1\times\cdots\times S_n$ such that $G(\hat\Omega_1,\dots,\hat\Omega_n)<0$. If $(\Omega_1,\dots,\Omega_n)\in\Lambda$ is a proper $\mathbf{R}^p_+$-solution of problem (P1), then there exist $\lambda=(\lambda_1,\dots,\lambda_p)\in\operatorname{int}\mathbf{R}^p_+$, $\beta=(\beta_1,\dots,\beta_m)\in\mathbf{R}^m_+$ and* *for each* $i = 1, \dots, p$, $k = 1, \dots, m$, *there exist* $(f^{1j}, \dots, f^{nj}) \in \partial F_j(\Omega_1, \dots, \Omega_n)$, $(g^{1k}, \dots, g^{nk}) \in \partial G_k(\Omega_1, \dots, \Omega_n)$ *such that*

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f^{ij} + \sum_{j=1}^{m} \beta_j g^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geq 0,$$

for all $(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$, *and*

$$\sum_{j=1}^{m} \beta_j G_j(\Omega_1, \dots, \Omega_n) = 0.$$

THEOREM 5. *(Duality theorem). Suppose that* F *and* G *are proper convex set functions on a convex subfamily* $S_1 \times \cdots \times S_n$ *of* Γ^n. *Suppose that* $F_1, \dots, F_p$ *are* w^**-continuous and* $G_1, \dots, G_m$*, except possibly one, are* w^**-continuous on* $S_1 \times \cdots \times S_n$ *and for each* $i = 1, \dots, n$, $\overline{S_i}$ *contains a relative interior point. Suppose further that there exists* $(\hat{\Omega}_1, \dots, \hat{\Omega}_n) \in S_1 \times \cdots \times S_n$*, such that* $G(\hat{\Omega}_1, \dots, \hat{\Omega}_n) < 0$. *If* $(\Omega_1, \dots, \Omega_n)$ *is a proper* $\mathbf{R}^p_+$*-solution of problem (P1), then there exist* $\lambda^* \in \operatorname{int} \mathbf{R}^p_+$, $\beta^* \in \mathbf{R}^m_+$ *such that* $((\Omega_1, \dots, \Omega_n), \lambda^*, \beta^*)$ *is a Pareto optimal solution of Problem (D1).*

PROOF. Since $(\Omega_1, \dots, \Omega_n)$ is a proper $\mathbf{R}^p_+$-solution of problem (P1), it follows from Lemma 4 that there exists $\lambda^* = (\lambda^*_1, \dots, \lambda^*_p) \in \operatorname{int} \mathbf{R}^p_+$ and $\beta^* = (\beta^*_1, \dots, \beta^*_m) \in \mathbf{R}^m_+$ and for each $j = 1, \dots, p$, $k = 1, \dots, m$, there exists $(f^{1j}, \dots, f^{nj}) \in \partial F_j(\Omega_1, \dots, \Omega_n)$, $(g^{1k}, \dots, g^{nk}) \in \partial G_k(\Omega_1, \dots, \Omega_n)$, such that

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f^{ij} + \sum_{j=1}^{m} \beta_j g^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geq 0,$$

for all $(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$, and

$$\sum_{j=1}^{m} \beta_j G_j(\Omega_1, \dots, \Omega_n) = 0.$$

In other words, $((\Omega_1, \dots, \Omega_n), \lambda^*, \beta^*)$ is feasible for (D1). Let $((\Lambda_1, \dots, \Lambda_n), \lambda, \beta)$ be any feasible solution for (D1). Then $\lambda = (\lambda_1, \dots, \lambda_p) \in \operatorname{int} \mathbf{R}^p_+$, and $\beta \in \mathbf{R}^p_+$, without loss of generality, we may assume that $\sum_{i=1}^{p} \lambda_i = 1$. Since $\langle \beta^*, G(\Omega_1, \dots, \Omega_n) \rangle = 0$, it follows from Theorem 1 that

$$\begin{aligned} \langle \lambda, F(\Omega_1, \dots, \Omega_n) + \langle \beta^*, G(\Omega_1, \dots, \Omega_n) \rangle e \rangle &= \langle \lambda, F(\Omega_1, \dots, \Omega_n) \rangle \\ &\geq \langle \lambda, F(\Lambda_1, \dots, \Lambda_n) + \langle \beta, G(\Lambda_1, \dots, \Lambda_n) \rangle e \rangle. \end{aligned}$$

Since $\lambda \in \operatorname{int} \mathbf{R}^p_+$, it follows from Lemma 2.2 that

$$F(\Omega_1, \dots, \Omega_n) + \langle \beta^*, G(\Omega_1, \dots, \Omega_n) \rangle e \in \bar{e}(B),$$

where

$$B = \{ F(\Lambda_1, \dots, \Lambda_n) + \langle \beta, G(\Lambda_1, \dots, \Lambda_n) \rangle e, \ ((\Lambda_1, \dots, \Lambda_n), \lambda, \beta) \text{ is feasible for (D1)} \}.$$

This shows that $((\Omega_1, \dots, \Omega_n), \lambda^*, \beta^*)$ is a Pareto optimal solution of (D1). ∎

COROLLARY 6. *Suppose that* $F : \Gamma^n \to \mathbf{R}^p$, $G : \Gamma^n \to \mathbf{R}^m$ *are convex and differentiable on* Γ^n *and there exists* $(\hat{\Omega}_1, \dots, \hat{\Omega}_n) \in \Gamma^n$ *such that* $G(\hat{\Omega}_1, \dots, \hat{\Omega}_n) < 0$. *If* $(\Omega_1, \dots, \Omega_n)$ *is a proper* $\mathbf{R}^p_+$*-solution of problem (P2), then there exists* $\lambda^* \in \operatorname{int} \mathbf{R}^p_+$, $\beta^* \in \mathbf{R}^m_+$*, such that* $((\Omega_1, \dots, \Omega_n), \lambda^*, \beta^*)$ *is a Pareto optimal solution of (D2).*

PROOF. If $F = (F_1, \ldots, F_p)$, $G = (G_1, \ldots, G_m)$ are differentiable on Γ^n, then it is easy to see that $F_1, \ldots, F_p$, $G_1, \ldots, G_m$ are w^*-continuous on Γ^n. It follows from Corollary 3.6 [3], that $\bar{\Gamma} = \{f \in L_1(X, \Gamma, \mu), 0 \le f \le 1\}$. Therefore $\bar{\Gamma}$ contains relative interior points and the corollary follows immediately from Theorem 5. ∎

LEMMA 7 [8]. *Let the set functions $F = (F_1, \ldots, F_p) : \Gamma^n \to \mathbf{R}^p$, and $G = (G_1, \ldots, G_m) : \Gamma^n \to \mathbf{R}^m$ be differentiable on Γ^n. Suppose that $(\Omega_1, \ldots, \Omega_n)$ is a Pareto optimal solution of (P1) and for each $S = 1, \ldots, p$, there exists $(\hat{\Omega}_1^s, \ldots, \hat{\Omega}_n^s) \in \Gamma^n$, such that*

$$G(\Omega_1, \ldots, \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{m} \langle g_*^{im}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle \right) < 0,$$

and for each $j = 1, \ldots, p$, $j \neq s$

$$\sum_{i=1}^{n} \langle f_*^{ij}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle > 0,$$

then there exists $\lambda = (\lambda_1, \ldots, \lambda_p) \in \operatorname{int} \mathbf{R}_+^p$, $\sum_{j=1}^{p} \lambda_j = 1$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbf{R}_+^m$, such that for each $i = 1, \ldots, n$,

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f_*^{ij} + \sum_{j=1}^{m} \beta_j g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \ge 0, \text{ for all } (\Lambda_1, \ldots, \Lambda_n) \in \Gamma^n,$$

$$\sum_{j=1}^{m} \beta_j G_j(\Omega_1, \ldots, \Omega_n) = 0, \quad G_j(\Omega_1, \ldots, \Omega_n) \le 0, \qquad j = 1, \ldots, m,$$

where f_*^{ij}, g_*^{ij} denote the i-th partial derivatives of F_j and G_j at $(\Omega_1, \ldots, \Omega_n)$, respectively.

THEOREM 8. *Let the set function $F = (F_1, \ldots, F_p) : \Gamma^n \to \mathbf{R}^p$, $G = (G_1, \ldots, G_m) : \Gamma^n \to \mathbf{R}^m$ be differentiable and convex on Γ^n. Suppose that $(\Omega_1, \ldots, \Omega_n)$ is a Pareto optimal solution of (P2) and for each $S = 1, \ldots, p$, there exists $(\hat{\Omega}_1^s, \ldots, \hat{\Omega}_n^s) \in \Gamma^n$, such that*

$$G(\Omega_1, \ldots, \Omega_n) + \left(\sum_{i=1}^{n} \langle g_*^{i1}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g_*^{im}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle \right) < 0,$$

and for each $j = 1, \ldots, p$, $j \neq s$,

$$\sum_{i=1}^{n} \langle f_*^{ij}, \chi_{\hat{\Omega}_i^s} - \chi_{\Omega_i} \rangle < 0.$$

Then there exists $\lambda^* \in \operatorname{int} \mathbf{R}_+^p$, $\beta^* \in \mathbf{R}_+^m$, such that $((\Omega_1, \ldots, \Omega_n), \lambda^*, \beta^*)$ is a Pareto optimal solution of (D2).

PROOF. It follows from Lemma 7 that there exists $\lambda^* \in \operatorname{int} \mathbf{R}_+^p$, $\beta^* \in \mathbf{R}_+^m$, such that $((\Omega_1, \ldots, \Omega_n), \lambda^*, \beta^*)$ is feasible for (D2) and $\sum_{j=1}^{m} \beta_j^* G_j(\Omega_1, \ldots, \Omega_n) = 0$, $\lambda^* = (\lambda_1^*, \ldots, \lambda_p^*)$, $\beta^* = (\beta_1^*, \ldots, \beta_m^*)$, $\sum_{j=1}^{p} \lambda_j^* = 1$. Applying Corollary 3 and following the similar arguments of Theorem 5, we complete the proof of this theorem. ∎

THEOREM 9. *(Converse duality). Let $S_1 \times \cdots \times S_n$ be a convex subfamily of Γ^n, $F : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ be convex and w^*-continuous, and $G : S_1 \times \cdots \times S_n \to \mathbf{R}^m$ be convex. Let $((\Lambda_1, \ldots, \Lambda_n), \lambda, \beta)$ be a feasible solution for (D1). Suppose that there exist $(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) \in S_1 \times \cdots \times S_n$ and a feasible solution $(\Omega_1, \ldots, \Omega_n)$ of (P1) such that $G(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) < 0$ and $\langle \lambda, F(\Omega_1, \ldots, \Omega_n) \rangle = \langle \lambda, F(\Lambda_1, \ldots, \Lambda_n) + \langle \beta, G(\Lambda_1, \ldots, \Lambda_n) \rangle e \rangle$.*

Then $(\Omega_1, \dots, \Omega_n)$ is a Geoffrion properly efficient solution of (P1).

PROOF. It follows from Theorem 1 that

$$\langle \lambda, F(B_1, \dots, B_n)\rangle \geq \langle \lambda, F(\Lambda_1, \dots, \Lambda_n) + \langle \lambda, G(\Lambda_1, \dots, \Lambda_n)\rangle e\rangle = \langle \lambda, F(\Omega_1, \dots, \Omega_n)\rangle,$$

for any feasible solution $(B_1, \dots, B_n)$ of problem (P1). Since $\lambda \in \operatorname{int} \mathbf{R}^p_+$, the theorem follows immediately from Lemmas 2.9, 2.14, and 2.15. ∎

In the following theorem, we shall show the weak duality, duality and converse duality theorems of the Mond-Weir type of dual problem.

THEOREM 10. *(Weak duality). Let $F : S_1 \times \cdots \times S_n \to \mathbf{R}^p$, $G : S_1 \times \cdots \times S_n \to \mathbf{R}^m$ be convex on a convex subfamily $S_1 \times \cdots \times S_n$ of Γ^n. Suppose that $(\Omega_1, \dots, \Omega_n)$ is feasible for (P1), $((\Lambda_1, \dots, \Lambda_n), \lambda, \beta)$ is feasible for (D3), then*

$$\langle \lambda, F(\Omega_1, \dots, \Omega_n)\rangle \geq \langle \lambda, F(\Lambda_1, \dots, \Lambda_n)\rangle.$$

PROOF. Since $((\Lambda_1, \dots, \Lambda_n), \lambda, \beta)$ is feasible for (D3), it follows that $\lambda = (\lambda_1, \dots, \lambda_p) \in \operatorname{int} \mathbf{R}^p_+$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbf{R}^m_+$, $\langle \lambda, G(\Lambda_1, \dots, \Lambda_n)\rangle \geq 0$ and for each $j = 1, \dots, p$, $i = 1, \dots, m$, there exists

$$(f^{1j}, \dots, f^{nj}) \in \partial F_i(\Lambda_1, \dots, \Lambda_n), \quad (g^{1i}, \dots, g^{ni}) \in \partial G_i(\Lambda_1, \dots, \Lambda_n),$$

such that

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f^{ij} + \sum_{j=1}^{m} \beta_j g^{ij}, \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \geq 0,$$

for all $(B_1, \dots, B_n) \in S_1 \times \cdots \times S_n$. Since $(\Omega_1, \dots, \Omega_n)$ is feasible for (P1), we have

$$0 \geq \langle \lambda, G(\Omega_1, \dots, \Omega_n)\rangle - \langle \lambda, G(\Lambda_1, \dots, \Lambda_n)\rangle.$$

Then by the definition of subdifferential, we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} \lambda_j \langle g^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle &= \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} \langle g^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle \\ &\leq \sum_{j=1}^{m} \lambda_j \left[G_j(\Omega_1, \dots, \Omega_n) - G_j(\Lambda_1, \dots, \Lambda_n)\right] \leq 0.\end{aligned}$$

Therefore

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i} \right\rangle \geq 0.$$

Again by the definition of subdifferential, we obtain

$$\begin{aligned}\langle \lambda, F(\Omega_1, \dots, \Omega_n)\rangle - \langle \lambda, F(\Lambda_1, \dots, \Lambda_n)\rangle &= \sum_{j=1}^{p} \lambda_j \left[F_j(\Omega_1, \dots, \Omega_n) - F_j(\Lambda_1, \dots, \Lambda_n)\right] \\ &\geq \sum_{j=1}^{p} \lambda_j \sum_{i=1}^{n} \langle f^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{p} \lambda_j \langle f^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle \geq 0.\end{aligned}$$

We complete the proof of the theorem. ∎

If we assume that F and G are differentiable on Γ^n, then the assumption that F and G are convex on Γ^n can be weakened by other conditions, in order to obtain similar results as Theorem 10, we need some definitions and results.

DEFINITION 3.1. *A set function $H : S_1 \times \cdots \times S_n \to \mathbf{R}$ is called quasiconvex on a convex subfamily $S_1 \times \cdots \times S_n$ of Γ^n if for each $(\Lambda_1, \dots, \Lambda_n)$, $(\Omega_1, \dots, \Omega_n)$ in $S_1 \times \cdots \times S_n$, $\lambda \in [0, 1]$*

and for each $i = 1, \dots, n$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ such that $(V_1^k(\lambda), \dots, V_n^k(\lambda)) \in S_1 \times \cdots \times S_n$ for all $k \in \mathbf{N}$ and $\overline{\lim}_{k\to\infty} H(V_1^k(\lambda), \dots, V_n^k(\lambda)) \leq \max\{H(\Omega_1, \dots, \Omega_n), H(\Lambda_1, \dots, \Lambda_n)\}$. A set function $F = (F_1, \dots, F_p) : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ is called quasiconvex on a convex subfamily $S_1 \times \cdots \times S_n$ of Γ^n, if for each $i = 1, \dots, p$, F_i is quasiconvex on $S_1 \times \cdots \times S_n$.

LEMMA 11 [8]. *Let $S_1 \times \cdots \times S_n$ be a nonempty convex subfamily of Γ^n and $F = (F_1, \dots, F_p) : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ be differentiable and quasiconvex on $S_1 \times \cdots \times S_n$. If for any $(\Omega_1, \dots, \Omega_n)$, $(\Lambda_1, \dots, \Lambda_n) \in S_1 \times \cdots \times S_n$, with $F(\Lambda_1, \dots, \Lambda_n) \leq F(\Omega_1, \dots, \Omega_n)$, then*

$$\left(\sum_{i=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle, \dots, \sum_{i=1}^{n} \langle f_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle\right) \leq 0,$$

where f_*^{ij} is as defined in Remark 2.

DEFINITION 3.2. *Let $S_1 \times \cdots \times S_n$ be a nonempty subfamily of Γ^n and let $F = (F_1, \dots, F_p) : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ be differentiable on $S_1 \times \cdots \times S_n$. The set function F is said to be pseudoconvex on $S_1 \times \cdots \times S_n$ if for each $(\Omega_1, \dots, \Omega_n)$, $(\Lambda_1, \dots, \Lambda_n)$ in $S_1 \times \cdots \times S_n$ with*

$$\left(\sum_{j=1}^{n} \langle f_*^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle, \dots, \sum_{i=1}^{p} \langle f_*^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle\right) \geq 0,$$

we have $F(\Lambda_1, \dots, \Lambda_n) \geq F(\Omega_1, \dots, \Omega_n)$, where f_*^{ij} is as in Remark 2.

THEOREM 12. *(Weak duality). Let $G : \Gamma^n \to \mathbf{R}^m$ be differentiable and quasiconvex and $F : \Gamma^n \to \mathbf{R}^p$ be pseudoconvex. If $(\Omega_1, \dots, \Omega_n)$ is feasible for problem (P2) and $((\Lambda_1, \dots, \Lambda_n), \lambda, \beta)$ is feasible for (D4). Then*

$$\langle \lambda, F(\Omega_1, \dots, \Omega_n)\rangle \geq \langle \lambda, F(\Lambda_1, \dots, \Lambda_n)\rangle.$$

PROOF. Since $((\Lambda_1, \dots, \Lambda_n), \lambda, \beta)$ is feasible for (D4), it follows that $\lambda = (\lambda_1, \dots, \lambda_p)) \in \operatorname{int} \mathbf{R}_+^p$, $\beta = (\beta_1, \dots, \beta_m) \in \mathbf{R}_+^m$ such that

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f_{\Lambda_1,\dots,\Lambda_n}^{ij} + \sum_{j=1}^{m} \beta_j g_{\Lambda_1,\dots,\Lambda_n}^{ij}, \chi_{B_i} - \chi_{\Lambda_i} \right\rangle \geq 0,$$

for all $(B_1, \dots, B_n) \in S_1 \times \cdots \times S_n$ and $\langle \lambda, G(\Lambda_1, \dots, \Lambda_n)\rangle \geq 0$. Since $(\Omega_1, \dots, \Omega_n)$ is feasible for (P2), we have

$$\langle \lambda, G(\Omega_1, \dots, \Omega_n)\rangle - \langle \lambda, G(\Lambda_1, \dots, \Lambda_n)\rangle \leq 0.$$

By Lemma 11, we obtain

$$\sum_{j=1}^{m} \lambda_j \sum_{i=1}^{n} \langle g_{\Lambda_1,\dots,\Lambda_n}^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle \leq 0.$$

Therefore

$$\sum_{j=1}^{p} \lambda_j \sum_{i=1}^{n} \langle f_{\Lambda_1,\dots,\Lambda_n}^{ij}, \chi_{\Omega_i} - \chi_{\Lambda_i}\rangle \geq 0.$$

Then by the pseudoconvexity of F, we obtain

$$\langle \lambda, F(\Omega_1, \dots, \Omega_n)\rangle \geq \langle \lambda, F(\Lambda_1, \dots, \Lambda_n)\rangle. \qquad \blacksquare$$

THEOREM 13. *(Duality theorem). Under the assumptions of Theorem 5, if $(\Omega_1, \dots, \Omega_n)$ is a proper $\mathbf{R}_+^p$-solution of problem (P1), then there exists $\lambda^* \in \operatorname{int} \mathbf{R}_+^p$, $\beta \in \mathbf{R}_+^m$, such that $((\Omega_1, \dots, \Omega_n), \lambda^*, \beta^*)$ is a Pareto optimal solution of (D3).*

PROOF. Since $(\Omega_1, \ldots, \Omega_n)$ is a proper $\mathbf{R}^p_+$-solution, it follows from Lemma 4 that there exists $\lambda^* \in \operatorname{int} \mathbf{R}^p_+$, $\beta^* \in \mathbf{R}^m_+$, such that $((\Omega_1, \ldots, \Omega_n), \lambda^*, \beta^*)$ is a feasible solution of (D3). Hence, if $((\Lambda_1, \ldots, \Lambda_n), \lambda, \beta)$ is any feasible solution of (D3), then by Theorem 10, we have

$$\langle \lambda, F(\Omega_1, \ldots, \Omega_n) \rangle \geq \langle \lambda, F(\Lambda_1, \ldots, \Lambda_n) \rangle,$$

and the theorem follows immediately from Lemma 2.2. ∎

THEOREM 14. *In Theorem 8, if F and G being convex is replaced by the condition that F be quasiconvex on* Γ^n *and G be pseudoconvex on* Γ^n, *then there exists* $\lambda^* \in \operatorname{int} \mathbf{R}^p_+$, $\beta^* \in \mathbf{R}^m_+$, *such that* $((\Omega_1, \ldots, \Omega_n), \lambda^*, \beta^*)$ *is a Pareto optimal solution of (D4).*

PROOF. Applying Theorem 12 and following the similar argument of Theorem 13, we complete the proof of this theorem. ∎

THEOREM 15. *(Converse duality theorem). Let* $S_1 \times \cdots \times S_n$ *be a convex subfamily of* Γ^n *and* $F : S_1 \times \cdots \times S_n \to \mathbf{R}^p$ *be convex and* w^**-continuous, and* $G : S_1 \times \cdots \times \to \mathbf{R}^m$ *be convex. Let* $((\Lambda_1, \ldots, \Lambda_n), \lambda, \beta)$ *be a feasible solution of (D3). Suppose that there exists* $(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) \in S_1 \times \cdots \times S_n$ *and a feasible solution* $(\Omega_1, \ldots, \Omega_n)$ *of (P1), such that* $G(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) < 0$ *and*

$$\langle \lambda, F(\Omega_1, \ldots, \Omega_n) \rangle = \langle \lambda, F(\Lambda_1, \ldots, \Lambda_n) \rangle.$$

Then $(\Omega_1, \ldots, \Omega_n)$ *is a Geffrion properly efficient solution of problem (P1).*

PROOF. Applying Theorem 10 and following the argument of Theorem 9, we obtain Theorem 15. ∎

The proof of the following theorem is essentially the same as Theorem 9; we omit its proof.

THEOREM 16. *Let* $F : \Gamma^n \to \mathbf{R}^p$, $G : \Gamma^n \to \mathbf{R}^m$ *be differentiable and convex and* $((\Lambda_1, \ldots, \Lambda_n), \lambda, \beta)$ *be a feasible solution of (D2) (resp. (D4)). Suppose further that there exists* $(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) \in \Gamma^n$ *and a feasible solution* $(\Omega_1, \ldots, \Omega_n)$ *of (P2), such that* $G(\hat{\Omega}_1, \ldots, \hat{\Omega}_n) < 0$ *and*

$$\langle \lambda, F(\Omega_1, \ldots, \Omega_n) \rangle = \langle \lambda, F(\Lambda_1, \ldots, \Lambda_n) + \langle \beta, G(\Lambda_1, \ldots, \Lambda_n) \rangle e \rangle,$$

(resp. $\langle \lambda, F(\Omega_1, \ldots, \Omega_n) \rangle = \langle \lambda, F(\Lambda_1, \ldots, \Lambda_n) \rangle$*). Then* $(\Omega_1, \ldots, \Omega_n)$ *is a Geoffrion properly efficient solution of problem (P2).*

REFERENCES

1. H.C. Lai and L.J. Lin, Optimality for set functions with values in order vector spaces, *J. Optim. Theory and Appl.* **63**, 405–423 (1989).
2. K. Tanaka and Y. Maruyama, The multiobjective problem of set function, *J. Inform. Optim. Sci.* **5**, 293–306 (1984).
3. J.H. Chou, W.S. Hsia and T.Y. Lee, Epigraph of convex set functions, *J. Math. Anal. Appl.* **118**, 247–254 (1986).
4. H.C. Lai and L.J. Lin, Moreau-Rockafellar type theorem of convex set functions, *J. Math. Anal. Appl.* **132**, 558–571 (1988).
5. T. Tanino and Y. Sawaragi, Duality theory in multiobjective programming, *J. Optim. Theory and Appl.* **27**, 509–529 (1979).
6. L.J. Lin, Optimality of differentiable vector-valued n-set functions, *J. Math. Anal. Appl.* **149**, 255–270 (1990).
7. J.H. Chou, W.S. Hsia and T.Y. Lee, Proper D-solution of multiobjective programming with set functions, *J. Optim. Theory and Appl.* **53**, 247–258 (1987).
8. L.J. Lin, On the optimality conditions of vector-valued n-set functions, *J. Math. Anal. Appl.*, to appear.
9. L.J. Lin, Convex n-set functions, preprint.
10. J.H. Chou, W.S. Hsia and T.Y. Lee, On multiple objective programming problems with set functions, *J. Math. Anal. Appl.* **105**, 383–394 (1985).
11. H.W. Corley, Optimization for n-set functions, *J. Math. Anal. Appl.* **127**, 193–205 (1987).
12. R.R. Egudo and M.A. Hanson, Multiobjective duality with invexity, *J. Math. Anal. Appl.* **126**, 469–477 (1987).
13. H.C. Lai and L.J. Lin, The Fenchel-Moreau theorem for set functions, *Proc. Amer. Math. Soc.* **103**, 85–90 (1988).

14. L.J. Lin, On the optimality of differentiable nonconvex n-set functions, preprint.
15. H.C. Lai, S.S. Yang and G.R. Hwang, Duality in mathematical programming of set functions—On Fenchel duality theorem, *J. Math. Anal. Appl.* **95**, 223–234 (1983).
16. H.C. Lai and S.S. Yang, Saddle point and duality in the optimization theory of convex set functions, *J. Austral. Math. Soc.* (Series B) **24**, 130–137 (1982).
17. R.J.T. Morris, Optimality constrained selection of measurable subset, *J. Math. Anal. Appl.* **70**, 543–562 (1979).
18. W. Rudin, *Functional Analysis*, McGraw-Hill, New York, (1969).