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On the Optimality Conditions of Vector-Valued n-Set Functions*

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In a finite atomless measure space (X, Γ, μ), the optimization problem of vector-valued n-set functions defined on a convex subfamily S of $\Gamma^n = \Gamma \times \cdots \times \Gamma$ is investigated. The necessary and sufficient conditions of Pareto optimal solution or proper $\mathbb{R}^p_+$-solution of optimization problem with differentiable vector valued n-set functions are given. © 1991 Academic Press, Inc.

1. INTRODUCTION

The general theory for optimizing set functions was first developed by Morris [12]. This type of problem arises in various areas and has many interesting applications in mathematics, engineering, and statistics, for example, in fluid flow, electrical insulator design, optimal plasma confinement (see Ref. [12]), and Neyman–Pearson lemma of statistics (see Ref. [3]). There are many results on the optimization problem of set functions, one can consult Refs. [12, 1, 2, 4–10, 14]. All the previous results on this type of problem are only confined to set functions of a single set. Corley [3] started to give the concepts of partial derivatives and derivatives of real-valued n-set functions and developed the general theory of n-set functions. In [7], we study the vector valued n-set functions optimization problem. This paper is a continuous work of [7]. Throughout this paper, we assume that (X, Γ, μ) is a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable. For any $n \in N$, we let $\mathbb{R}^n$ be the n-dimensional Euclidean space. We also let $S \subset \Gamma^n = \Gamma \times \cdots \times \Gamma$ be a subfamily of Γ^n and $F\colon S \to \mathbb{R}^p$, $H\colon S \to \mathbb{R}^r$, and $G\colon S \to \mathbb{R}^m$ be vector-valued n-set functions defined on S.

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We consider two optimization problems as

$$\begin{aligned}&\text{minimize } F(A_1, \ldots, A_n)\\&\text{subject to } (A_1, \ldots, A_n) \in S,\ G(A_1, \ldots, A_n) \leqq 0,\ H(A_1, \ldots, A_n) = 0,\end{aligned} \tag{P}$$

and

$$\begin{aligned}&\text{minimize } F(A_1, \ldots, A_n)\\&\text{subject to } (A_1, \ldots, A_n) \in \Gamma^n,\ G(A_1, \ldots, A_n) \leqq 0.\end{aligned} \tag{P1}$$

In [7], we define the derivative of vector-valued n-set functions, we establish the necessary and sufficient conditions for the existence of a weak local minimum to problem (P1) in terms of the partial derivative of vector-valued n-set functions involved. This paper is a continuous work of [7]. The sufficient conditions of Pareto optimal solution to problem (P) and the necessary conditions of pareto optimal solution of (P1) with non-convex differentiable n-set functions are developed. The necessary and sufficient conditions of proper $\mathbb{R}^p_+$-solution to problem (P1) with convex differentiable vector-valued n-set function are also derived.

2. Preliminaries

We define a pseudometric d on $\Gamma^n = \Gamma \times \cdots \times \Gamma = \{(A_1, \ldots, A_n) \mid A_i \in \Gamma, i = 1, 2, \ldots, n\}$ as

$$d[(\Omega_1, \ldots, \Omega_n), (A_1, \ldots, A_n)] = \left\{\sum_{i=1}^{n} [\mu(\Omega_i \Delta A_i)]^2\right\}^{1/2},$$

where $(\Omega_1, \ldots, \Omega_n)$, $(A_1, \ldots, A_n) \in \Gamma^n$ and $\Omega_i \Delta A_i$ denote the symmetric difference for Ω_i and A_i. For $f \in L_1(X, \Gamma, \mu)$ and $\Omega \in \Gamma$, the integral $\int_\Omega f\, d\mu$ will be denoted by $\langle f, \chi_\Omega \rangle$, where χ_Ω denotes the characteristic function of Ω.

Definition 2.1. A set function F: $\Gamma \to \mathbb{R}$ is said to be differentiable at $\Omega \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$ the derivative of F at Ω such that

$$F(A) = F(\Omega) + \langle f, \chi_A - \chi_\Omega \rangle + \mu(\Omega \Delta A)\, E(\Omega, A),$$

where $\lim_{\mu(\Omega \Delta A) \to 0} E(\Omega, A) = 0$.

We define the partial derivatives of n-set functions.

Definition 2.2. Let F: $\Gamma^n \to \mathbb{R}$ and $(\Omega_1, \ldots, \Omega_n) \in \Gamma^n$. Then F is said to have partial derivative with respect to A_i if the set function

$$H(A_i) = F(\Omega_1, \ldots, \Omega_{i-1}, A_i, \Omega_{i+1}, \ldots, \Omega_n)$$

has derivative h_{Ω_i} at Ω_i. In this case we define the ith partial derivative of F at $(\Omega_1, ..., \Omega_n)$ to be $f^i_{\Omega_1,...,\Omega_n} = h_{\Omega_i}$.

Using the partial derivative of n-set function, we can define the derivative of vector-valued n-set functions.

DEFINITION 2.3 [7]. Let $S \subset \Gamma^n$, $F = (F_1, ..., F_m)$: $S \to \mathbb{R}^m$, and $(\Omega_1, ..., \Omega_n) \in S$. Then F is said to be differentiable at $(\Omega_1, ..., \Omega_n)$ if the partials $f^{ij}_{\Omega_1,...,\Omega_n}$, $i = 1, 2, ..., n$, of F_j exist for each $j = 1, 2, ..., m$ and satisfy

$$\begin{aligned} &F(\Lambda_1, ..., \Lambda_n) \\ &\quad = F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f^{i1}_{\Omega_1,...,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{im}_{\Omega_1,...,\Omega_m}, \chi_{\Lambda_1} - \chi_{\Omega_i} \rangle \right) \\ &\qquad + W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)), \end{aligned}$$

for all $(\Lambda_1, ..., \Lambda_n) \in S$, where

$$\frac{W_F((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))}{d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n))} \to 0$$

as $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) \to 0$. If F is differentiable at every point $(\Omega_1, ..., \Omega_n)$ of S, we say that F is differentiable on S.

Throughout the paper if $F = (F_1, ..., F_p)$: $S \subset \Gamma^n \to \mathbb{R}^p$ $G = (G_1, ..., G_m)$: $S \to \mathbb{R}^m$ and $H = (H_1, ..., H_r)$: $S \to \mathbb{R}^r$ are differentiable at $(\Omega_1, ..., \Omega_n)$, we will denote f^{ij}, g^{ij}, and h^{ij} the ith partial derivatives of F_j, G_j, and H_j at $(\Omega_1, ..., \Omega_n)$ respectively.

For two vectors $x = (x_1, ..., x_p)$ and $y = (y_1, ..., y_p)$ in p-dimensional Euclidean space $\mathbb{R}^p$, we introduce the following notations

(1) $x < y$ iff $x_i < y_i$ for all $i = 1, 2, ..., p$.

(2) $x \leqslant y$ iff $x_i \leqslant y_i$ for all $i = 1, 2, ..., p$ and $x \neq y$.

(3) $x \leqq y$ iff $x_i \leqslant y_i$ for all $i = 1, 2, ..., p$.

The nonnegative orthant and the nonpositive orthant in $\mathbb{R}^p$ are denoted by

$$\mathbb{R}^p_+ = \{x \in \mathbb{R}^p; x \leqq 0\} \qquad \text{and} \qquad \mathbb{R}^p_- = \{x \in \mathbb{R}^p; x \leqq 0\},$$

respectively, where 0 is the zero vector $(0, 0, ..., 0)$ in $\mathbb{R}^p$. We also denote $\langle x, y \rangle = \sum_{i=1}^p x_i y_i$ as the inner product of $x = (x_1, ..., x_p)$ and $y = (y_1, ..., y_p)$ in $\mathbb{R}^p$. For a set $E \subset \mathbb{R}^p$, the set of all interior points of E will be denoted by int E and the set of closure of E in $\mathbb{R}^p$ will be denoted by $\bar{E}$.

DEFINITION 2.4. A set $E \subset \mathbb{R}^p$ is said to be $\mathbb{R}^p_+$-convex if $E + \mathbb{R}^p_+$ is a convex set in $\mathbb{R}^p$.

DEFINITION 2.5. A point x^* is a lower efficient point of $E \subset \mathbb{R}^p$ if $x^* \in E$ and there is no $x \in E$ such that $x \leqslant x^*$. We denote the set of all lower efficient points of E by $\underset{\sim}{e}(E)$.

LEMMA 2.6 [14]. *Suppose that for a point $x^* \in E \subseteq \mathbb{R}^p$, there exists a $\hat{\mu} \in \operatorname{int} \mathbb{R}^p_+$ such that $\langle \hat{\mu}, x^* \rangle \leqslant \langle \hat{\mu}, x \rangle$ for $x \in E$. Then $x^* \in \underset{\sim}{e}(E)$.*

DEFINITION 2.7. A point $x^* \in \mathbb{R}^p$ is said to be a properly efficient point of $E \subset \mathbb{R}^p$ if $x^* \in \underset{\sim}{e}(E)$ and $\overline{E + \mathbb{R}^p_+ - x^*} \cap \mathbb{R}^p_- = \{0\}$.

DEFINITION 2.8. Given a p-dimensional vector-valued function $f = (f_1, ..., f_p)$: $X \to \mathbb{R}^p$, $f_i \in L_1(X, \Gamma, \mu)$, $i = 1, ..., p$, we say that f separates $\Omega \in \Gamma$ if $(\langle f_1, \chi_\Omega \rangle, ..., \langle f_p, \chi_\Omega \rangle)$ is a properly efficient point of the set $Y = \{(\langle f_1, \chi_\Lambda \rangle, ..., \langle f_p, \chi_\Lambda \rangle);\ \Lambda \in \Gamma\}$.

It follows from [12, Proposition 3.2 and Lemma 3.3], for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exist sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Lambda \setminus \Omega} \qquad \text{and} \qquad x_{\Lambda_m} \xrightarrow{w^*} (1-\lambda) \chi_{\Omega \setminus \Lambda} \tag{1}$$

implies

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Lambda + (1-\lambda) \chi_\Omega \tag{2}$$

where w^* stands for the w^*-convergence. The sequence $\{V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ satisfying (1) and (2) is called the Morris sequence associated with $(\Omega, \Lambda, \lambda)$.

DEFINITION 2.9. A subfamily S of Γ^n is convex if given $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n) \in S$, and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$, where N is the set of natural numbers.

DEFINITION 2.10. A set function $F = (F_1, ..., F_n)$: $S \to \mathbb{R}^p$ is called convex on a convex subfamily S of Γ^n if for each $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in S$, $\lambda \in [0, 1]$ there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\begin{aligned}
&\overline{\lim_{k \to \infty}}\ F(V_1^k(\lambda), ..., V_n^k(\lambda)) \\
&\quad = \Big(\overline{\lim_{k \to \infty}}\ F_1(V_1^k(\lambda), ..., V_n^k(\lambda)), ..., \overline{\lim_{k \to \infty}}\ F_p(V_1^k(\lambda), ..., V_n^k(\lambda))\Big) \\
&\quad \leqq \lambda F(\Lambda_1, ..., \Lambda_n) + (1-\lambda)\, F(\Omega_1, ..., \Omega_n).
\end{aligned}$$

LEMMA 2.11 [15, Lemma 2.4]. *Let E be a $\mathbb{R}_+^p$-convex set. Then $y_0 \in E$ satisfies*

$$\overline{[E+\mathbb{R}_+^p - y_0]} \cap \mathbb{R}_-^p = \{0\}$$

iff there exists a vector $\mu \in \operatorname{int} \mathbb{R}_+^p$ such that

$$\langle \mu, y_0 \rangle \leqslant \langle \mu, y \rangle \qquad \text{for any} \quad y \in E.$$

LEMMA 2.12 (Liapunov [13]). *Let $f_1, ..., f_p \in L_1(X, \Gamma, \mu)$, then the set $\{(\langle f_1, \chi_\Lambda \rangle, ..., \langle f_p, \chi_\Lambda \rangle), \Lambda \in \Gamma\}$ is convex and compact.*

3. Main Results

Throughout this paper, we will denote $A = \{(\Lambda_1, ..., \Lambda_n) \in \Gamma^n, G(\Lambda_1, ..., \Lambda_n) \leqq 0\}$, $A' = \{(\Lambda_1, ..., \Lambda_n) \in \Gamma^n, G(\Lambda_1, ..., \Lambda_n) < 0\}$, and $\hat{A} = \{(\Lambda_1, ..., \Lambda_n) \in S, G(\Lambda_1, ..., \Lambda_n) \leqq 0, H(\Lambda_1, ..., \Lambda_n) = 0\}$. The following lemma follows immediately from the definition of convex subfamily and properties of Morris sequence.

LEMMA 3.1. *Let $(\Omega_1, ..., \Omega_n) \in \Gamma^n$ and $G: \Gamma^n \to \mathbb{R}^m$ be convex, then for each $\delta > 0$ the set*

$$\begin{aligned} &B_\delta((\Omega_1, ..., \Omega_n)) \\ &\quad = \{(\Lambda_1, ..., \Lambda_n) \in \Gamma^n; d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta, G(\Lambda_1, ..., \Lambda_n) < 0\} \end{aligned}$$

is a convex subfamily of Γ^n.

Proof. Suppose $(\Lambda_1, ..., \Lambda_n), (\hat{\Omega}_1, ..., \hat{\Omega}_n) \in B_\delta((\Omega_1, ..., \Omega_n))$ and $\lambda \in [0, 1]$. Then $(\Lambda_1, ..., \Lambda_n)$, $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$, $G(\Lambda_1, ..., \Lambda_n) < 0$, $G(\hat{\Omega}_1, ..., \hat{\Omega}_n) < 0$, $d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta$, $d((\hat{\Omega}_1, ..., \hat{\Omega}_n), (\Omega_1, ..., \Omega_n)) < \delta$, and for each $i = 1, 2, ..., n$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\hat{\Omega}_i, \Lambda_i \lambda)$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \Gamma^n$ for all $k \in N$. Since

$$\begin{aligned} &\lim_{k \to \infty} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \\ &\quad = \lim_{k \to \infty} \left\{ \sum_{i=1}^n \|\chi_{V_i^k(\lambda)} - \chi_{\Omega_i}\|_{L^1}^2 \right\}^{1/2} \\ &\quad = \left(\sum_{i=1}^n \|\lambda \chi_{\Lambda_i} + (1-\lambda) \chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L^1}^2 \right)^{1/2} \\ &\quad = \left\{ \sum_{i=1}^n \|\lambda(\chi_{\Lambda_i} - \chi_{\Omega_i}) + (1-\lambda)(\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L^1}^2 \right\}^{1/2} \end{aligned}$$

$$\leqslant \lambda \left(\sum_{i=1}^{n} \|\chi_{A_i} - \chi_{\Omega_i}\|_{L^1}^2 \right)^{1/2} + (1-\lambda) \left(\sum_{i=1}^{n} \|\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L^1}^2 \right)^{1/2}$$
$$= \lambda \left(\sum_{i=1}^{n} [\mu(A_i \Delta \Omega_i)]^2 \right)^{1/2} + (1-\lambda) \left(\sum_{i=1}^{n} [\mu(\hat{\Omega}_i \Delta \Omega_i)]^2 \right)^{1/2}$$
$$= \lambda d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) + (1-\lambda)\, d((\hat{\Omega}_1, ..., \hat{\Omega}_n), (\Omega_1, ..., \Omega_n))$$
$$< \lambda\delta + (1-\lambda)\delta = \delta.$$

Hence there exists a natural number M_1 such that

$$d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < \delta \qquad \text{for all} \quad k \geqslant M_1, \tag{3}$$

since G is convex,

$$\overline{\lim_{k \to \infty}}\, G(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqq \lambda G(A_1, ..., A_n) + (1-\lambda)\, G(\hat{\Omega}_1, ..., \hat{\Omega}_n) < 0.$$

Therefore, there exists a natural number M_2 such that

$$G(V_1^k(\lambda), ..., V_n^k(\lambda)) < 0. \tag{4}$$

Let $M = \max\{M_1, M_2\}$, then from (3) and (4), we see that if $k \geqslant M$,

$$d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < \delta \qquad \text{and} \qquad G(V_1^k(\lambda), ..., V_n^k(\lambda)) < 0.$$

Thus $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in B_\delta((\Omega_1, ..., \Omega_n))$ for $k \geqslant M$. This shows that $B_\delta((\Omega_1, ..., \Omega_n))$ is a convex subfamily of Γ^n.

COROLLARY 3.2. *Let $(\Omega_1, ..., \Omega_n) \in \Gamma^n$ and $G: \Gamma^n \to \mathbb{R}^m$ be a convex set function, then the set A' is a convex subfamily of Γ^n.*

Proof. It is easy to see that $A' = \bigcup_{m=1}^{\infty} B_m((\Omega_1, ..., \Omega_n))$ where

$$B_m((\Omega_1, ..., \Omega_n))$$
$$= \{(A_1, ..., A_n) \in \Gamma^n; d((A_1, ..., A_n), (\Omega_1, ..., \Omega_n)) < m \text{ and } G(A_1, ..., A_n) < 0\}$$

and the corollary follows immediately from Lemma 3.1.

For any $S \subset \Gamma$, we denote $\bar{S}$ the w^*-closure of $\chi_S = \{\chi_A; A \in S\}$ in $L_\infty(X, \Gamma, \mu)$, then $\bar{\Gamma} = \{f \in L_\infty(X, \Gamma, \mu); 0 \leqslant f \leqslant 1\}$ [1, Corollary 3.6]. For $f \in \bar{\Gamma}$, we denote $N(f)$ the family of all w^*-neighborhood of f in $\bar{\Gamma}$. Since $\bar{\Gamma}$ is w^*-compact and $L_1(X. \Gamma, \mu)$ is separable, $\bar{\Gamma}$ is metrizable [1]. Therefore $\bar{\Gamma} \times \cdots \times \bar{\Gamma} = (\bar{\Gamma})^n$ is also metrizable.

LEMMA 3.3 [7]. *Let $F=(F_1, ..., F_p)$: $\Gamma^n \to \mathbb{R}^p$ be differentiable and convex on Γ^n, then for all $(A_1, ..., A_n)$, $(\Omega_1, ..., \Omega_n) \in \Gamma^n$*

$$F(A_1, ..., A_n) \geqq F(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right).$$

A set function F: $S \to \mathbb{R}^p$ is said to be w^*-continuous at $(\Omega_1, ..., \Omega_n) \in S$, if for any sequence $\{(\Omega_1^k, ..., \Omega_n^k)\}$ in S and for each $i=1, ..., n$, $\chi_{\Omega_i^k} \xrightarrow{w^*} \chi_{\Omega_i}$ as $k \to \infty$ implies $F(\Omega_1, ..., \Omega_n) = \lim_{k \to \infty} F(\Omega_1^k, ..., \Omega_n^k)$. F is said to be w^*-continuous on S, if F is w^*-continuous at each point $(\Omega_1, ..., \Omega_n) \in S$.

LEMMA 3.4. *Let S be a convex subfamily of Γ^n and $F: S \to \mathbb{R}^p$ be a w^*-continuous and convex set function. Then the set $\overline{F(S)}$ is $\mathbb{R}^p_+$-convex.*

Proof. The proof of this lemma is similar to Lemma 3.1 of [2].

LEMMA 3.5. *Let F: $\Gamma^n \to \mathbb{R}^p$ be w^*-continuous and G: $\Gamma^n \to \mathbb{R}^m$ be convex. Suppose that there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$ such that $G(\hat{\Omega}_1, ..., \hat{\Omega}_n) < 0$. Then $\overline{F(A)} = \overline{F(A')}$ and $\overline{F(A)}$ is $\mathbb{R}^p_+$-convex.*

Proof. Since $(\hat{\Omega}_1, ..., \hat{\Omega}_n) < 0$, A' is not empty. Let $(\Omega_1, ..., \Omega_n) \in A'$ and $(A_1, ..., A_n) \in A$, for each $i=1, ..., n$ and each positive integer m, let $\{V^i_{m,k}\}$ be the Morris sequence in Γ such that

$$\chi_{V^i_{m,k}} \xrightarrow[\text{as } k \to \infty]{w^*} \frac{1}{m} \chi_{\Omega_i} + \left(1 - \frac{1}{m}\right) \chi_{A_i}.$$

By the convexity of G,

$$\overline{\lim_{k \to \infty}}\ G((V^1_{m,k}, ..., V^n_{m,k})) \leqq \frac{1}{m} G((\Omega_1, ..., \Omega_n)) + \left(1 - \frac{1}{m}\right) G((A_1, ..., A_n)) < 0.$$

Thus there exists a natural number M such that

$$G((V^1_{m,k}, ..., V^n_{m,k})) < 0 \qquad \text{for} \quad k \geqslant M.$$

This shows that $(V^1_{m,k}, ..., V^n_{m,k}) \in A'$ for $k \geqslant M$. Hence

$$(\chi_{V^1_{m,k}}, ..., \chi_{V^n_{m,k}}) \in \chi_{A'} = \{(\chi_{B_1}, ..., \chi_{B_n}), (B_1, ..., B_n) \in A'\} \subset \bar{\Gamma} \times \cdots \times \bar{\Gamma}.$$

We note that $(\chi_{A_1}, ..., \chi_{A_n})$ is a cluster point of $\{(\chi_{V^1_{m,k}}, ..., \chi_{V^n_{m,k}}), m, k \in N\}$. Since $\bar{\Gamma}$ is metrizable in the w^*-topology and $\bar{\Gamma} \times \cdots \times \bar{\Gamma}$ is metrizable, there exists a subsequence $\{(V^1_l, ..., V^n_l)\}$ of $\{(V^1_{m,k}, ..., V^n_{m,k})\}$ such that $\chi_{V^i_l} \xrightarrow[\text{as } l \to \infty]{w^*} \chi_{A_i}$ for each $i=1, ..., n$. Since F is w^*-continuous, we have

$$\lim_{l \to \infty} F((V^1_l, ..., V^n_l)) = F((A_1, ..., A_n)).$$

This shows that $F(A_1, ..., A_n) \in \overline{F(A')}$. Therefore $\overline{F(A)} = \overline{F(A')}$ and the lemma follows immediately from Lemma 3.4 and Corollary 3.2.

We say $(\Omega_1, ..., \Omega_n) \in \hat{A}$ (resp. $(\Omega_1, ..., \Omega_n) \in A$) is a Pareto optimal solution to problem (P) (resp. (P1)) if

$$F(\Omega_1, ..., \Omega_n) \in \underline{e}(\hat{A}) \text{ (resp. } F(\Omega_1, ..., \Omega_n) \in \underline{e}(A)).$$

DEFINITION 3.6. $(\Omega_1, ..., \Omega_n) \in A$ is said to be a proper $\mathbb{R}^p_+$-solution of (P1) if $\overline{F(A) + \mathbb{R}^p_+ - F(\Omega_1, ..., \Omega_n)} \cap \mathbb{R}^p_- = \{0\}$.

LEMMA 3.7. *Suppose that S is a nonempty subfamily of Γ^n, $F = (F_1, ..., F_p)$: $S \to \mathbb{R}^p$, and $\overline{F(S)}$ is $\mathbb{R}^p_+$ — convex. Then $(\Omega_1, ..., \Omega_n) \in S$ is a proper $\mathbb{R}^p_+$-solution of* (P1) *if and only if $(\Omega_1, ..., \Omega_n)$ is optimal for* (P1(λ)) *for some $\lambda \in \operatorname{int} \mathbb{R}^p_+$, where*

$$\begin{aligned} &\min \sum_{i=1}^{p} \lambda_i F_i(A_1, ..., A_n) \\ &\textit{subject to } (A_1, ..., A_n) \in S. \end{aligned} \qquad (\mathrm{P1}(\lambda))$$

Proof. The proof of this lemma is the same as Theorem 3.1 of [2].

THEOREM 3.8. *Let $F = (F_1, ..., F_p)$: $\Gamma^n \to \mathbb{R}^p$ be w^*-continuous and convex and G: $\Gamma^n \to \mathbb{R}^m$ be convex. Suppose that there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$ such that $G(\hat{\Omega}_1, ..., \hat{\Omega}_n) < 0$. Then $(\Omega_1, ..., \Omega_n) \in A$ is a proper $\mathbb{R}^p_+$-solution if and only if $(\Omega_1, ..., \Omega_n)$ is optimal for* (MP1(λ)) *for some $\lambda = (\lambda_1, ..., \lambda_n) \in \operatorname{int} \mathbb{R}^p_+$ where*

$$\begin{aligned} &\min \sum_{i=1}^{p} \lambda_i F_i(A_1, ..., A_n) \\ &\textit{subject to } (A_1, ..., A_n) \in A. \end{aligned} \qquad (\mathrm{MP1}(\lambda))$$

Proof. This theorem follows immediately from Lemmas 3.5 and 3.7.

DEFINITION 3.9. A point $(\Omega_1, ..., \Omega_n) \in \Gamma^n$ is said to be a local minimum to problem (P1) if there exists $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) \leqq F(A_1, ..., A_n)$ for all $(A_1, ..., A_n) \in \Gamma^n$, $G(A_1, ..., A_n) \leqq 0$ satisfying $d[(A_1, ..., A_n), (\Omega_1, ..., \Omega_n)] < \delta$.

LEMMA 3.10 [3, Corollary 3.9]. *In problem* (P1) *if $F: \Gamma^n \to \mathbb{R}^1$ and $G = (G_1, ..., G_m)$: $\Gamma^n \to \mathbb{R}^m$ are differentiable at $(\Omega_1, ..., \Omega_n) \in \Gamma^n$. Suppose that $(\Omega_1, ..., \Omega_n)$ is a local minimum to problem* (P1) *and that there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$ such that*

$$G_j(\Omega_1, ..., \Omega_n) + \sum_{i=1}^{n} \langle g^{ij}_{\Omega_1, ..., \Omega_n}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle < 0, \qquad j = 1, ..., m. \qquad (5)$$

Then there exist $\lambda_1, ..., \lambda_n$ *such that*

$$\left\langle f^i + \sum_{j=1}^{m} \lambda_j g^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geqslant 0 \tag{6}$$

for all $\Lambda_i \in \Gamma$, $i = 1, ..., n$.

$$\lambda_j G_j(\Omega_1, ..., \Omega_n) = 0 \tag{7}$$

$$\lambda_1, ..., \lambda_m \geqslant 0 \tag{8}$$

$$G_j(\Omega_1, ..., \Omega_n) \leqslant 0, \qquad j = 1, ..., m, \tag{9}$$

where $g^{ij}_{\hat{\Omega}_1, ..., \hat{\Omega}_n}$ *is the* i*th partial derivative of* G_j *at* $(\hat{\Omega}_1, ..., \hat{\Omega}_n)$.

THEOREM 3.11 (Necessary and Sufficient Conditions for Constrained Local Minimum). *In problem* (P1), *if* $F = (F_1, ..., F_p)$: $\Gamma^n \to \mathbb{R}^p$ *and* $G = (G_1, ..., G_m)$: $\Gamma^n \to \mathbb{R}^m$ *are convex on* Γ^n *and differentiable at* $(\Omega_1, ..., \Omega_n) \in \Gamma^n$, *suppose that* $(\Omega_1, ..., \Omega_n)$ *is a proper* $\mathbb{R}^p_+$*-solution of problem* (P1). *Suppose further that there exist* $(\hat{\Omega}_1, ..., \hat{\Omega}_n)$ *and* $(B_1, ..., B_n) \in \Gamma^n$ *such that*

$$G(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle g^{i1}_{\hat{\Omega}_1, ..., \hat{\Omega}_n}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{im}_{\hat{\Omega}_1, ..., \hat{\Omega}_n}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle \right) < 0 \tag{10}$$

and

$$G(B_1, ..., B_n) < 0. \tag{11}$$

Then there exist $\lambda = (\lambda_1, ..., \lambda_n) \in \operatorname{int} \mathbb{R}^p_+$, $\mu = (\mu_1, ..., \mu_m) \in \mathbb{R}^m_+$ *such that*

$$\left\langle \sum_{j=1}^{p} \lambda_j f^{ij} + \sum_{j=1}^{m} \mu_j g^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geqslant 0, \qquad i = 1, ..., m, \Lambda_i \in \Gamma, \tag{12}$$

$$\mu_j G_j(\Omega_1, ..., \Omega_n) = 0, \qquad j = 1, ..., m \tag{13}$$

$$G_j(\Omega_1, ..., \Omega_n) \leqslant 0, \qquad j = 1, ..., m. \tag{14}$$

Conversely, if there exist $\lambda = (\lambda_1, ..., \lambda_p) \in \operatorname{int} \mathbb{R}^p_+$, $\mu = (\mu_1, ..., \mu_m) \in \mathbb{R}^m_+$, *and* $(B_1, ..., B_n) \in \Gamma^n$ *such that* (11), (12), (13), *and* (14) *hold, then* $(\Omega_1, ..., \Omega_n)$ *is a proper* $\mathbb{R}^p_+$*-solution of problem* (P1).

Proof. Since $(\Omega_1, ..., \Omega_n)$ is a proper $\mathbb{R}^p_+$-solution of problem (P1), it follows from Theorem 3.8 that there exists $\lambda = (\lambda_1, ..., \lambda_p) \in \operatorname{int} \mathbb{R}^p_+$ such that

$$\langle \lambda, F(\lambda_1, ..., \Lambda_n) \rangle \geqslant \langle \lambda, F(\Omega_1, ..., \Omega_n) \rangle$$

for all $(\Lambda_1, ..., \Lambda_n) \in A$. Then by Lemma 3.10, there exists $\mu = (\mu_1, ..., \mu_m) \in \mathbb{R}^m_+$ such that (12), (13), (14) are true.

Conversely, if there exist $\lambda=(\lambda_1, ..., \lambda_p)\in \operatorname{int} \mathbb{R}^p_+$, $\mu=(\mu_1, ..., \mu_m)\in \mathbb{R}^m_+$ and $(B_1, ..., B_n)\in \Gamma^n$ such that (11), (12), (13), and (14) hold. Since F and G are differentiable and convex on Γ^n, it follows from Lemma 3.3 that for all $(\Lambda_1, ..., \Lambda_n)\in \Gamma^n$

$$F(\Lambda_1, ..., \Lambda_n)-F(\Omega_1, ..., \Omega_n)$$
$$\geqq\left(\sum_{i=1}^n \langle f^{i1}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle, ..., \sum_{i=1}^n \langle f^{ip}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle\right) \tag{15}$$
$$G(\Lambda_1, ..., \Lambda_n)-G(\Omega_1, ..., \Omega_n)$$
$$\geqq\left(\sum_{i=1}^n \langle g^{i1}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle, ..., \sum_{i=1}^n \langle g^{im}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle\right) \tag{16}$$

Since $\lambda\in \operatorname{int} \mathbb{R}^p_+$, $\mu\in\mathbb{R}^m_+$, it follows from (15), 16), and (12) that

$$\begin{aligned}
&\langle \lambda, F(\Lambda_1, ..., \Lambda_n)-F(\Omega_1, ..., \Omega_n)\rangle+\langle \mu, G(\Lambda_1, ..., \Lambda_n)-G(\Omega_1, ..., \Omega_n)\rangle\\
&\geqslant\left\langle \lambda, \left(\sum_{i=1}^n \langle f^{i1}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle, ..., \sum_{i=1}^n \langle f^{ip}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle\right)\right\rangle\\
&\quad+\left\langle \mu, \left(\sum_{i=1}^n \langle g^{i1}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle, ..., \sum_{i=1}^n \langle g^{im}, \chi_{\Lambda_i}-\chi_{\Omega_i}\rangle\right)\right\rangle\\
&=\sum_{i=1}^n \left\langle \sum_{j=1}^p \lambda_j f^j+\sum_{j=1}^m \mu_j g^{ij}, \chi_{\Lambda_i}-\chi_{\Omega_i}\right\rangle\geqslant 0.
\end{aligned}$$

As $\mu\in\mathbb{R}^m_+$, $\mu_j G_j(\Omega_1, ..., \Omega_n)=0$, $j=1, ..., m$, we have

$$\begin{aligned}
&\langle \lambda, F(\Lambda_1, ..., \Lambda_n)-F(\Omega_1, ..., \Omega_n)\rangle\\
&\quad\geqslant \langle \lambda, F(\Lambda_1, ..., \Lambda_n)-F(\Omega_1, ..., \Omega_n)\rangle+\langle \mu, G(\Lambda_1, ..., \Lambda_n)-G(\Omega_1, ..., \Omega_n)\rangle\\
&\quad\geqslant 0.
\end{aligned}$$

For any $(\Lambda_1, ..., \Lambda_n)\in A$, $(\Omega_1, ..., \Omega_n)$ is a proper $\mathbb{R}^p_+$-solution follows immediately from Theorem 3.8.

Remark 3.12. In Theorem 3.11, if the condition

$$\left\langle \sum_{j=1}^p \lambda_j f^{ij}+\sum_{j=1}^m \mu_j g^{ij}, \chi_{\Lambda_i}-\chi_{\Omega_i}\right\rangle\geqslant 0$$

for all $i=1, ..., n$ and all $\Lambda_i\in\Gamma$ is replaced by

$$\sum_{i=1}^n\left\langle \sum_{j=1}^p \lambda_j f^{ij}+\sum_{j=1}^m \mu_j g^{ij}, \chi_{\Lambda_i}-\chi_{\Omega_i}\right\rangle\geqslant 0$$

for all $(\Lambda_1, ..., \Lambda_n)\in\Gamma^n$, we see that the theorem is still true.

As a consequence of Theorem 3.11, we have the following two theorems.

THEOREM 3.13. *In problem* (P1), *let* $F=(F_1, ..., F_p)$: $\Gamma^n \to \mathbb{R}^p$, *and* $G=(G_1, ..., G_m)$: $\Gamma^n \to \mathbb{R}^m$ *be convex on* Γ^n *and differentiable at* $(\Omega_1, ..., \Omega_n) \in \Gamma^n$. *Suppose that* $(\Omega_1, ..., \Omega_n)$ *is a proper* $\mathbb{R}^p_+$*-solution of problem* (P1). *Suppose further that there exist* $(\hat{\Omega}_1, ..., \hat{\Omega}_n)$ *and* $(B_1, ..., B_n) \in \Gamma^n$ *such that*

$$G(\Omega_1, ..., \Omega_n)+\left(\sum_{i=1}^{n} \langle g^{i1}_{\hat{\Omega}_1,...,\hat{\Omega}_n}, \chi_{\hat{\Omega}_i}-\chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle g^{im}_{\hat{\Omega}_1,...,\hat{\Omega}_n}, \chi_{\hat{\Omega}_i}-\chi_{\Omega_i}\rangle\right)<0$$

and $G(B_1, ..., B_n)<0$, *then there exists* $\mu=(\mu_1, ..., \mu_m) \in \mathbb{R}^m_+$ *such that for each* $i=1, ..., n$,

$$\left(f^{i1}+\sum_{j=1}^{m} \mu_j g^{ij}, ..., f^{ip}+\sum_{j=1}^{m} \mu_j g^{ij}\right) \text{ separates } \Omega_i, \tag{17}$$

$$\mu_j G_j(\Omega_n, ..., \Omega_n)=0, \qquad j=1, ..., m \tag{18}$$

$$G_j(\Omega_1, ..., \Omega_n) \leqslant 0, \qquad j=1, ..., m, \tag{19}$$

where $g^{ij}_{\hat{\Omega}_1,...,\hat{\Omega}_n}$ *denotes ith partial derivative of* G_j *at* $(\hat{\Omega}_1, ..., \hat{\Omega}_n)$.

Proof. It follows from Theorem 3.11, there exist $\lambda=(\lambda_1, ..., \lambda_p) \in \operatorname{int} \mathbb{R}^p_+$, $\mu=(\mu_1, ..., \mu_m) \in \in \mathbb{R}^m_+$ such that (12), (13), and (14) are true. Without loss of generality, we may assume that $\sum_{j=1}^{p} \lambda_j=1$. In view of (12), we have for each $i=1, 2, ..., n$,

$$\langle \lambda, (\langle f^{i1}, \chi_{A_i}-\chi_{\Omega_i}\rangle, ..., \langle f^{ip}, \chi_{A_i}-\chi_{\Omega_i}\rangle)\rangle+\left\langle \sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i}-\chi_{\Omega_i}\right\rangle \geqslant 0 \quad \text{for all } A_i \in \Gamma. \tag{20}$$

Since $\sum_{i=1}^{n} \lambda_i=1$, it follows from (20) that

$$\langle \lambda, (\langle f^{i1}, \chi_{A_i}-\chi_{\Omega_i}\rangle, ..., \langle f^{ip}, \chi_{A_i}-\chi_{\Omega_i}\rangle) + \left(\left\langle \sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i}-\chi_{\Omega_i}\right\rangle, ..., \left\langle \sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i}-\chi_{\Omega_i}\right\rangle\right)\right\rangle \geqslant 0 \quad \text{for all } A_i \in \Gamma.$$

Therefore for each $i=1, ..., n$ and $(A_1, ..., A_n) \in \Gamma^n$,

$$\left\langle \lambda, \left(\left\langle f^{i1}+\sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i}\right\rangle\right)\right\rangle \geqslant \left\langle \lambda, \left(\left\langle f^{i1}+\sum_{j=1}^{m} \mu_j g^{ij}, \chi_{\Omega_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m} \mu_j g^{ij}, \chi_{\Omega_i}\right\rangle\right)\right\rangle. \tag{21}$$

Then by Lemma 2.6 and (21),

$$\left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle\right)\in \underline{e}(Y_i),$$

where

$$Y_i=\left\{\left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}\right\rangle\right); \Lambda_i\in\Gamma\right\}.$$

Since Y_i is convex by Liapunov's Lemma (Lemma 2.12), we get by (21) and Lemma 2.11 that

$$\left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle\right)$$

is a properly efficient point of Y_i for each $i=1, ..., n$.

This shows that for each $i=1, ..., n$

$$\left(f^{i1}+\sum_{j=1}^{m}\lambda_j g^{ij}, ..., f^{ip}+\sum_{j=1}^{m}\lambda_j g^{ij}\right) \quad \text{separates } \Omega_i$$

and the proof of the theorem is completed.

The following theorem gives the sufficient conditions for the existence of the proper $\mathbb{R}^p_+$-solution.

THEOREM 3.14. *In problem* (P1) *if F and G are differentiable and convex on Γ^n. Suppose that there exist $(B_1, ..., B_n)\in\Gamma^n$, $\lambda=(\lambda_1, ..., \lambda_p)\in\operatorname{int}\mathbb{R}^p_+$, $\mu=(\mu_1, ..., \mu_m)\in\mathbb{R}^m_+$ such that* (11), (17), (18), *and* (19) *hold, then $(\Omega_1, ..., \Omega_n)\in\Gamma^n$ is a proper $\mathbb{R}^p_+$-solution of problem* (P1).

Proof. By Lemmas 2.11 and 2.12, there exists $\lambda=(\lambda_1, ..., \lambda_p)\in\operatorname{int}\mathbb{R}^p_+$ such that for each $i=1, ..., n$ and for all $\Lambda_i\in\Gamma$,

$$\left\langle\lambda, \left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}\right\rangle\right)\right\rangle$$
$$\geqslant\left\langle\lambda, \left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Omega_i}\right\rangle\right)\right\rangle$$

or

$$\left\langle\lambda, \left(\left\langle f^{i1}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}-\chi_{\Omega_i}\right\rangle, ..., \left\langle f^{ip}+\sum_{j=1}^{m}\mu_j g^{ij}, \chi_{\Lambda_i}-\chi_{\Omega_i}\right\rangle\right)\right\rangle\geqslant 0 \quad (22)$$

for all $i=1, \ldots, n$ and $A_i \in \Gamma$. Without loss of generality, we may assume that $\sum_{j=1}^{p} \lambda_j = 1$. From (22) and $\sum_{j=1}^{p} \lambda_j = 1$, we see that

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} \lambda_j f^{ij} + \sum_{j=1}^{m} \mu_j g^{ij}, \chi_{A_i} - \chi_{\Omega_i} \right\rangle \geqslant 0$$

for all $(A_1, \ldots, A_n) \in \Gamma^n$. By Theorem 3.11 and Remark 3.12, we complete the proof of the theorem.

DEFINITION 3.15. A set function $F: S \to \mathbb{R}$ is called quasiconvex on a convex subfamily S of Γ^n if for each $(\Omega_1, \ldots, \Omega_n)$, $(A_1, \ldots, A_n)$ in S, $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with (Ω_i, A_i, λ) for each $i=1, \ldots, n$ such that $(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \leqslant \max\{F(\Omega_1, \ldots, \Omega_n), F(A_1, \ldots, A_n)\}.$$

DEFINITION 3.16. A set function $F=(F_1, \ldots, F_p)$: $S \to \mathbb{R}^p$ is called quasiconvex on a convex subfamily S of Γ^n, if for each $i=1, \ldots, p$, F_i is quasiconvex on S.

Remark. It is easy to see that if a set function is convex, then it is quasiconvex, but the converse is not true, in [8], we give an example of a quasiconvex set function which is not convex.

LEMMA 3.17. *Let S be a nonempty convex subfamily of Γ^n and $F=(F_1, \ldots, F_p)$: $S \to \mathbb{R}^p$ be differentiable and quasiconvex on S. If for any $(\Omega_1, \ldots, \Omega_n)$, $(A_1, \ldots, A_n) \in S$ with $F(A_1, \ldots, A_n) \leqq F(\Omega_1, \ldots, \Omega_n)$ then*

$$\left(\sum_{i=1}^{m} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{m} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \leqq 0.$$

Proof. Since F is quasiconvex on S, it follows that F_j is quasiconvex on S for each $j=1, \ldots, n$. Let $\lambda \in (0, 1)$, then there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with (Ω_i, A_i, λ) for each $i=1, \ldots, n$ such that $(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F_j(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \leqslant F_j(\Omega_1, \ldots, \Omega_n).$$

Since F is differentiable at $(\Omega_1, \ldots, \Omega_n) \in S$, it follows that

$$\begin{aligned} &F_j(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \\ &\quad = F_j(\Omega_1, \ldots, \Omega_n) + \sum_{i=1}^{n} \langle f^{ij}, \chi_{V_i^k(\lambda)} - \chi_{\Omega_i} \rangle \\ &\qquad + d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) \cdot E((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)), \end{aligned}$$

where

$$E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \to 0$$
$$\text{as} \quad d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \to 0.$$

In theorem 3 of [7], we show that

$$\overline{\lim_{k\to\infty}}\, d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n))$$
$$\cdot E((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \in o(\lambda).$$

Hence

$$\overline{\lim_{k\to\infty}}\, F_j(V_1^k(\lambda), ..., V_n^k(\lambda))$$
$$= F_j(\Omega_1, ..., \Omega_n) + \lambda \sum_{i=1}^n \langle f^{ij}, \chi_{A_i} - \chi_{\Omega_i}\rangle + o(\lambda)$$
$$\leqslant F_j(\Omega_1, ..., \Omega_n).$$

That is,

$$\lambda \sum_{i=1}^n \langle f^{ij}, \chi_{A_i} - \chi_{\Omega_i}\rangle + o(\lambda) \leqslant 0, \qquad \text{for all} \quad j = 1, ..., p.$$

Dividing both sides of the above inequality by λ and letting $\lambda \to 0$, we have

$$\sum_{i=1}^n \langle f^{ij}, \chi_{A_i} - \chi_{\Omega_i}\rangle \leqslant 0, \qquad \text{for all} \quad j = 1, ..., p.$$

It follows that

$$\left(\sum_{i=1}^n \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^n \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i}\rangle\right) \leqq 0.$$

The following theorem gives sufficient conditions for existence of a Pareto optimal solution to problem (P) with convex objective function and non-convex constrained functions.

THEOREM 3.18. *In problem* (P), *if* S *is a convex subfamily of* Γ^n *and* $(\Omega_1, ..., \Omega_n) \in S$. *Suppose that*

(i) *F, G, and H are differentiable at* $(\Omega_1, ..., \Omega\text{-}_n)$.

(ii) *F: $S \to \mathbb{R}^p$ is a convex set function.*

(iii) $G_I = (G_{s_1}, ..., G_{s_j})$ *and* $H = (H_1, ..., H_r)$ *are quasiconvex on* S, *where* $I = \{i;\ G_i(\Omega_1, ..., \Omega_n) = 0\} = \{s_1, ..., s_j\}$.

(iv) *There exists* $u \in \operatorname{int} \mathbb{R}^p_+$, $v_I \in \mathbb{R}^j_+$, $w \in \mathbb{R}^r_+$, *such that*

$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle v_I, \left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{is_j}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle w, \left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \geqslant 0.$$

(v) $G(\Omega_1, ..., \Omega_n) \leqq 0$.

(vi) $H(\Omega_1, ..., \Omega_n) = 0$.

Then $(\Omega_1, ..., \Omega_n)$ *is a Pareto optimal solution to problem* (P).

Proof. Suppose that $(\Omega_1, ..., \Omega_n)$ is not a Pareto optimal solution to problem (P). Then there exists $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$ such that

$$F(\Lambda_1, ..., \Lambda_n) - F(\Omega_1, ..., \Omega_n) \leqslant 0,$$
$$G(\Lambda_1, ..., \Lambda_n) \leqq 0,$$
$$H(\Lambda_1, ..., \Lambda_n) = 0.$$

Hence

$$G_I(\Lambda_1, ..., \Lambda_n) \leqq G_I(\Omega_1, ..., \Omega_n) = 0,$$
$$H(\Lambda_1, ..., \Lambda_n) = H(\Omega_1, ..., \Omega_n) = 0.$$

By the convexity of F and quasiconvexity of G_I and H, Lemmas 3.3 and 3.18, we have

$$\left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right)$$
$$\leqq F(\Lambda_1, ..., \Lambda_n) - F(\Omega_1, ..., \Omega_n) \leqslant 0, \tag{23}$$

$$\left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{is_j}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \leqq 0, \tag{24}$$

$$\left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \leqq 0. \tag{25}$$

Since $u>0$, $v \geqq 0$, $w \geqq 0$, it follows from (23), (24), (25) that we have

$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle v_I, \left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{is_j}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle w, \left(\sum_{i=1}^{n} \langle h_{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle < 0.$$

This inequality contradicts hypothesis (iv). Hence $(\Omega_1, ..., \Omega_n)$ is a Pareto optimal solution to problem (P).

DEFINITION 3.19. Let S be a nonempty subfamily of Γ^n and let $F = (F_1, ..., F_p)$: $S \to \mathbb{R}^p$ be differentiable on S. The set function F is said to be pseudoconvex on S if for each $(\Omega_1, ..., \Omega_n)$ and $(A_1, ..., A_n)$ in S, with

$$\left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \geqq 0$$

we have

$$F(A_1, ..., A_n) \geqq F(\Omega_1, ..., \Omega_n).$$

Remark. It follows from Lemma 3.3 that if F is a convex set function, then it is pseudoconvex, but the converse is not true. In [8], we give an example to show that a pseudoconvex set function is not convex.

THEOREM 3.20. *In problem* (P), *suppose that*

(i) *F, G_I, and H are differentiable at $(\Omega_1, ..., \Omega_n) \in S \subset \Gamma^n$, where*

$$I = \{i; G_i(\Omega_1, ..., \Omega_n) = 0\} = \{s_1, ..., s_j\}.$$

(ii) *There exist $u \in \operatorname{int} \mathbb{R}^p_+$, $v \in \mathbb{R}^j_+$, and $w \in \mathbb{R}^r_+$ such that*

$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle v, \left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{is_j}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle$$
$$+ \left\langle w, \left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \geqslant 0 \quad (26)$$

for all $(A_1, ..., A_n) \in \hat{A}$.

(iii) $G(\Omega_1, ..., \Omega_n) \leqq 0$.

(iv) $H(\Omega_1, ..., \Omega_n) = 0$.

(v) $\sum_{i=1}^{p} u_i F_i + \sum_{i \in I} v_i G_i + \sum_{j=1}^{r} w_j H_j$ *is pseudoconvex on S.*

Then $(\Omega_1, ..., \Omega_n)$ *is a Pareto optimal solution to problem* (P).

Proof. Assume that $(\Omega_1, ..., \Omega_n)$ is not a Pareto optimal solution to problem (P), then there exists $(\Lambda_1, ..., \Lambda_n) \in \hat{A}$, $G(\Lambda_1, ..., \Lambda_n) \leqq 0$, $H(\Lambda_1, ..., \Lambda_n) = 0$ such that

$$F(\Lambda_1, ..., \Lambda_n) \leqslant F(\Omega_1, ..., \Omega_n).$$

By (i), (iv), we see that

$$G_I(\Lambda_1, ..., \Lambda_n) \leqq 0 = G_I(\Omega_1, ..., \Omega_n)$$

and

$$H(\Lambda_1, ..., \Lambda_n) = H(\Omega_1, ..., \Omega_n) = 0.$$

Since $u \in \operatorname{int} \mathbb{R}_+^p$, $v \in \mathbb{R}_+^j$, it follows that

$$\langle u, F(\Lambda_1, ..., \Lambda_n)\rangle + \langle v, G_I(\Lambda_1, ..., \Lambda_n)\rangle + \langle w, H(\Lambda_1, ..., \Lambda_n)\rangle$$
$$< \langle u, F(\Omega_1, ..., \Omega_n)\rangle + \langle v, G_I(\Omega_1, ..., \Omega_n)\rangle + \langle w, H(\Omega_1, ..., \Omega_n)\rangle.$$

By assumption, $\sum_{i=1}^{p} u_i F_i + \sum_{i \in I} v_i G_i + \sum_{j=1}^{r} w_j H_j$ is pseudoconvex at $(\Omega_1, ..., \Omega_n)$, we have

$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle\right)\right\rangle$$
$$+ \left\langle v, \left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle g^{is_j}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle\right)\right\rangle$$
$$+ \left\langle w, \left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{\Lambda_i} - \chi_{\Omega_i}\rangle\right)\right\rangle < 0 \quad (27)$$

for all $(\Lambda_1, ..., \Lambda_n) \in \hat{A}$. But (27) contradicts hypothesis (ii). Hence $(\Omega_1, ..., \Omega_n)$ is a Pareto optimal solution to problem (P).

THEOREM 3.21. *In problem* (P), *suppose that S is a convex subfamily of* Γ^n, $(\Omega_1, ..., \Omega_n) \in S$, *and*

(i) *F, G, and H are differentiable at* $(\Omega_1, ..., \Omega_n)$.

(ii) *There exist* $u \in \operatorname{int} \mathbb{R}^p_+$, $v \in \mathbb{R}^m_+$, $w \in \mathbb{R}^r_+$ *such that*

(a) $\sum_{i=1}^{p} u_i F_i$ *is pseudoconvex on* S,

(b) $\sum_{i \in I} v_i G_i$ *is quasiconvex on* S,

(c) $\sum_{i=1}^{r} w_i H_i$ *is quasiconvex on* S,

(d)
$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle f^{ip}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle + \left\langle v, \left(\sum_{i=1}^{n} \langle g^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g^{im}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle + \left\langle w, \left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle h^{ir}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \geqslant 0$$

for all $(\Lambda_1, \ldots, \Lambda_n) \in \hat{\Lambda}$.

(iii) $\langle v, G(\Omega_1, \ldots, \Omega_n) \rangle = 0$.

(iv) $G(\Omega_1, \ldots, \Omega_n) \leqq 0$.

(v) $H(\Omega_1, \ldots, \Omega_n) = 0$.

Then $(\Omega_1, \ldots, \Omega_n)$ is a Pareto optimal solution to problem (P).

Proof. Since $\langle v, G(\Omega_1, \ldots, \Omega_n) \rangle = 0$, $G(\Omega_1, \ldots, \Omega_n) \leqq 0$, $v \geqq 0$, it follows that

$$v_i G_i(\Omega_1, \ldots, \Omega_n) = 0 \qquad \text{for all } i.$$

Therefore

$$\sum_{i \in I} v_i G_i(\Omega_1, \ldots, \Omega_n) = 0.$$

For any $(\Lambda_1, \ldots, \Lambda_n) \in S$ with $G(\Lambda_1, \ldots, \Lambda_n) \leqq 0$, we have

$$\sum_{i \in I} v_i G_i(\Lambda_1, \ldots, \Lambda_n) \leqslant \sum_{i \in I} v_i G_i(\Omega_1, \ldots, \Omega_n).$$

Since $\sum_{i \in I} v_i G_i$ is quasiconvex on S, it follows that

$$\left\langle v_I, \left(\sum_{i=1}^{n} \langle g^{is_1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g^{is_j}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \leqslant 0 \tag{28}$$

for $(\Lambda_1, \ldots, \Lambda_n) \in \hat{\Lambda}$. As $v_j = 0$ for each $j \in \{1, \ldots, m\} \backslash I = (t_1, \ldots, t_l)$, we have

$$\left\langle v_{\{1, \ldots, m\} \backslash I}, \left(\sum_{i=1}^{n} \langle g^{it_1}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle, \ldots, \sum_{i=1}^{n} \langle g^{it_l}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right) \right\rangle = 0 \tag{29}$$

for all $(A_1, ..., A_n) \in \hat{A}$. Similarly $\sum_{j=1}^{r} w_j H_j(A_1, ..., A_n) = \sum_{j=1}^{r} w_j H_j(\Omega_1, ..., \Omega_n) = 0$ for all $(A_1, ..., A_n) \in \hat{A}$. As $\sum_{j=1}^{r} w_j H_j$ is quasiconvex on S, we have

$$\left\langle w, \left(\sum_{i=1}^{n} \langle h^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle h^{ir}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \leqslant 0 \qquad (30)$$

for all $(A_1, ..., A_n) \in \hat{A}$. By (ii)(d), (28), (29), and (30), we have

$$\left\langle u, \left(\sum_{i=1}^{n} \langle f^{i1}, \chi_{A_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle f^{ip}, \chi_{A_i} - \chi_{\Omega_i} \rangle \right) \right\rangle \geqslant 0$$

for all $(A_1, ..., A_n) \in \hat{A}$. Since $\sum_{i=1}^{p} u_i F_i$ is assumed to be pseudoconvex on S, we have

$$\langle u, F(A_1, ..., A_n) \rangle \geqslant \langle u, F(\Omega_1, ..., \Omega_m) \rangle$$

for all $(A_1, ..., A_n) \in \hat{A}$. For $u \in \operatorname{int} \mathbb{R}^p_+$, it follows from Lemma 2.6 that $(\Omega_1, ..., \Omega_n)$ is a Pareto optimal solution to problem (P).

LEMMA 3.22 [5]. *In problem* (P1), $(\Omega_1, ..., \Omega_n)$ *is a Pareto optimal solution if and only if* $(\Omega_1, ..., \Omega_n)$ *minimize each* F_j *on the constraint set*

$$\begin{aligned} C_j = \{ &(A_1, ..., A_n) \in \Gamma^n, F_i(A_1, ..., A_n) \leqslant F_i(\Omega_1, ..., \Omega_n), \\ &i \neq j \text{ and } G(A_1, ..., A_n) \leqslant 0 \}. \end{aligned} \qquad (31)$$

The following theorem establishes necessary conditions for a Pareto optimal solution of problem (P1) when the set functions are differentiable.

THEOREM 3.23. *Let the set functions* $F = (F_1, ..., F_p)$: $\Gamma^n \to \mathbb{R}^p$ *and* $G = (G_1, ..., G_m)$: $\Gamma^n \to \mathbb{R}^m$ *be differentiable on* Γ^n. *Suppose that* $(\Omega_1, ..., \Omega_n)$ *is a Pareto optimal solution of* (P1) *and for each* $s = 1, ..., p$ *there exist* $(\hat{\Omega}^s_1, ..., \hat{\Omega}^s_n) \in \Gamma^n$ *such that*

$$G(\Omega_1, ..., \Omega_n) + \left(\sum_{i=1}^{n} \langle g^{i1}_{\hat{\Omega}^s_1, ..., \hat{\Omega}^s_n}, \chi_{\hat{\Omega}^s_i} - \chi_{\Omega_i} \rangle, ..., \sum_{i=1}^{n} \langle g^{im}_{\hat{\Omega}^s_1, ..., \hat{\Omega}^s_n}, \chi_{\hat{\Omega}^s_i} - \chi_{\Omega_i} \rangle \right) < 0$$

and for each $j = 1, ..., p$, $j \neq s$

$$\left(\sum_{i=1}^{n} \langle f^{ij}_{\hat{\Omega}^s_1, ..., \hat{\Omega}^s_n}, \chi_{\hat{\Omega}^s_i} - \chi_{\Omega_i} \rangle \right) < 0,$$

then there exist $v = (v_1, ..., v_p) \in \operatorname{int} \mathbb{R}^p_+$, $\sum_{j=1}^{p} v_j = 1$, $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ *such that*

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} v_j f^{ij}_{\Omega_1, ..., \Omega_n} + \sum_{j=1}^{m} \lambda_j g^{ij}_{\Omega_1, ..., \Omega_n}, \chi_{A_i} - \chi_{\Omega_i} \right\rangle \geqslant 0 \qquad (32)$$

for all $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$

$$\sum_{j=1}^{m} v_j G_j(\Omega_1, ..., \Omega_n) = 0$$

$$G_j(\Omega_1, ..., \Omega_n) \geqslant 0, \qquad j = 1, ..., m,$$

where $f^{ij}_{\Lambda_1,...,\Lambda_n}$, $g^{ij}_{\Lambda_1,...,\Lambda_n}$ *are the ith partial derivatives of* F_j *and* G_j *at* $(\Lambda_1, ..., \Lambda_n)$, *respectively.*

Proof. Since $(\Omega_1, ..., \Omega_n)$ is a Pareto optimal solution of (P1), it follows from Lemma 3.23 that $(\Omega_1, ..., \Omega_n)$ minimizes each F_j on the constraint set C_j of (31). Then by Lemma 3.9 for each $i = 1, ..., n$, $j = 1, ..., p$, there exist $\beta_{1j}, ..., \beta_{mj}, \gamma_{1j}, ..., \gamma_{j-1,j}, \gamma_{j+1,j}, ..., \gamma_{pj}$ such that

$$\left\langle f^{ij}_{\Omega_1,...,\Omega_n} + \sum_{k=1}^{m} \beta_{kj} g^{ik}_{\Omega_1,...,\Omega_n} + \sum_{\substack{k=1 \\ k \neq j}}^{p} \gamma_{kj} f^{ik}_{\Omega_1,...,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geqslant 0 \qquad (33)$$

for all $\Lambda_i \in \Gamma$

$$\sum_{k=1}^{m} \beta_{kj} G_k(\Omega_1, ..., \Omega_n) = 0,$$

$$G_k(\Omega_1, ..., \Omega_n) \leqslant 0, \qquad k = 1, ..., m.$$

Letting $j = 1, ..., p$ in (33) and then summing up, we obtain

$$\left\langle \left(1 + \sum_{j=2}^{p} \gamma_{1j}\right) f^{i1}_{\Omega_1,...,\Omega_n} + \cdots + \left(1 + \sum_{j=1}^{p-1} \gamma_{pj}\right) f^{ip}_{\Omega_1,...,\Omega_n} + \sum_{j=1}^{p} \sum_{k=1}^{m} \beta_{kj} \langle g^{ik}_{\Omega_1,...,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \right\rangle \geqslant 0 \qquad \text{for all} \quad \Lambda_i \in \Gamma.$$

Letting

$$\mu_s = 1 + \sum_{\substack{j=1 \\ j \neq s}}^{p} \gamma_{sj}, \qquad v_j = \frac{\mu_j}{\sum_{j=1}^{p} \mu_j}, \qquad \lambda_k = \frac{\sum_{j=1}^{p} \beta_{kj}}{\sum_{j=1}^{p} \mu_j},$$

then $\sum_{j=1}^{p} v_j = 1$, $v = (v_1, ..., v_p) \in \operatorname{int} \mathbb{R}^p_+$, $\lambda = (\lambda_1, ..., \lambda_m) \in \mathbb{R}^m_+$ and for all $i = 1, ..., n$

$$\left\langle \sum_{j=1}^{p} v_j f^{ij}_{\Omega_1,...,\Omega_n} + \sum_{j=1}^{m} \lambda_j g^{ij}_{\Omega_1,...,\Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geqslant 0 \qquad \text{for all} \quad \Lambda_i \in \Gamma.$$

$$\sum_{j=1}^{m} \lambda_j G_j(\Omega_1, ..., \Omega_n) = \sum_{j=1}^{m} \sum_{i=1}^{p} \frac{\beta_{ji}}{\sum_{j=1}^{p} \mu_j} G_j(\Omega_1, ..., \Omega_n)$$

$$= \frac{1}{\sum_{j=1}^{p} \mu_j} \sum_{j=1}^{m} \sum_{i=1}^{p} \beta_{ji} G_j(\Omega_1, ..., \Omega_n) = 0.$$

Hence

$$\sum_{i=1}^{n} \left\langle \sum_{j=1}^{p} v_j f^{ij}_{\Omega_1, ..., \Omega_n} + \sum_{j=1}^{m} \lambda_j g^{ij}_{\Omega_1, ..., \Omega_n}, \chi_{A_i} - \chi_{\Omega_i} \right\rangle \geqslant 0$$

for all $(A_1, ..., A_n) \in \Gamma^n$. We complete the proof of the theorem.

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