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On the Optimality of Differentiable Nonconvex n-Set Functions*

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Our main contribution is the extension of the concepts of quasiconvexity and pseudoconvexity to n-set functions. Some properties of differentiable nonconvex n-set functions are established. Necessary and sufficient conditions for the existence of an optimal solution of the nonconvex program with n-set functions are characterized by derivatives of the n-set functions involved. A duality theorem for the nonconvex program with n-set functions is also developed in this paper. © 1992 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper let (X, Γ, μ) be a finite atomless measure space with $L_1(X, \Gamma, \mu)$ separable, and let $F, G_1, ..., G_m, H_1, ..., H_l$ be real-valued n-set functions defined on a convex subfamily S of $\Gamma^n = \Gamma \times \Gamma \times \cdots \times \Gamma$. Then we consider an optimization problem as

$$\begin{aligned} \text{Minimize:} \quad & F(\Omega_1, ..., \Omega_n) \\ \text{Subject to:} \quad & (\Omega_1, ..., \Omega_n) \in S \text{ and} \\ & G_i(\Omega_1, ..., \Omega_n) \leqslant 0, \; i = 1, 2, ..., m. \\ & H_j(\Omega_1, ..., \Omega_n) = 0, \; j = 1, 2, ..., l. \end{aligned} \tag{P}$$

This type of problem arises in various mathematical areas. For example, see the Neyman–Pearson lemma of statistics [20], which gives the sufficient condition for maximizing an integral over a single set. The necessity of this condition, and the existence of a solution were established in [8]. These results were generalized to n sets and a duality theory was developed in [5, 6]. However, all these results were for a special case for

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set functions involving integrals. Morris [18, 19] had first developed the general theory for optimizing set functions. Subsequent works [2–4, 9–13, 20] on the optimization problem are only confined to functions of a single set and the optimization problem does not have equality constraints and the set functions are convex. Corley [7] started to develop the general theory for n-set functions and gave the concepts of partial derivatives and the derivative of the n-set function. In this paper, we begin to give the concepts of pseudoconvexity and quasiconvexity of set functions, then we establish some properties of nonconvex, differentiable n-set functions. In Theorem 3.8, we show a sufficient condition for the existence of optimal solutions to problem (P) with equality constraints and nonconvex n-set functions. If the problem (P) does not have equality constraints and the set functions we consider are convex, then Theorem 3.8 reduces to Theorem 4.7 of [7]. A necessary condition for the existence of local minimum and a duality theorem for (P) with nonconvex n-set functions are also developed in this paper. Because the n-set functions are defined on a subfamily of a semialgebra rather than on a linear space, there are a good deal of differences between the optimization problem of nonconvex, differentiable n-set functions on a convex subfamily of a semialgebra and for usual functions on a linear space.

2. Preliminaries

Throughout the paper, let $\Gamma^n = \{(\Omega_1, ..., \Omega_n), \Omega_i \in \Gamma, i = 1, 2, ..., n\}$. As a matter of fact Γ^n is only a semialgebra but not a σ-algebra.

We defined a pseudometric d on the semialgebra Γ^n in the following way:

$$d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) = \left\{ \sum_{i=1}^{n} [\mu(\Omega_i \Delta \Lambda_i)]^2 \right\}^{1/2},$$

$\Omega_i, \Lambda_i \in \Gamma$, $i = 1, 2, ..., n$, where Δ denotes the symmetric difference. Each $\Omega \in \Gamma$ can be identified with its characteristic function $\chi_\Omega \in L_\infty(X, \Gamma, \mu) \subset L_1(X, \Gamma, \mu)$ and so that the σ-field Γ is identified as a subset $\chi_\Lambda = \{\chi_\Omega \mid \Omega \in \Gamma\}$ of $L_\infty(X, \Gamma, \mu)$. Essentially $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n)$ will be regarded as equivalent if $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) = 0$. We admit $F(\Omega_1, ..., \Omega_n = F(\Lambda_1, ..., \Lambda_n)$ if $d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) = 0$. For $f \in L_1(X, \Gamma, \mu)$ and $\Omega \in \Gamma$, the integral $\int_\Omega f\, d\mu$ will be denoted by $\langle f, \chi_\Omega \rangle$. Similar to [19, Proposition 3.2 and Lemma 3.3], for any $(\Omega, \Lambda, \lambda) \in \Gamma \times \Gamma \times [0, 1]$, there exist sequences $\{\Omega_n\}$ and $\{\Lambda_n\}$ in Γ such that

$$\chi_{\Omega_n} \xrightarrow{w^*} \lambda \chi_{\Lambda \setminus \Omega} \quad \text{and} \quad \chi_{\Lambda_n} \xrightarrow{w^*} (1-\lambda) \chi_{\Omega \setminus \Lambda} \tag{1}$$

imply

$$\chi_{\Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)} \xrightarrow{w^*} \lambda \chi_\Lambda + (1-\lambda)\chi_\Omega, \tag{2}$$

where w^* stands for the w^*-convergence. The sequence $\{V_n(\lambda) = \Omega_n \cup \Lambda_n \cup (\Omega \cap \Lambda)\}$ satisfying (1) and (2) is called the Morris sequence with $(\Omega, \Lambda, \lambda)$.

DEFINITION 2.1. A subfamily S of Γ^n is convex if given $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n)$ in S and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, 2, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$, for all $k \in N$, where N is the set of natural numbers.

EXAMPLE. *For a fix* $(\Omega_1, ..., \Omega_n) \in \Gamma^n$ *and* $\delta > 0$, *the subfamily* $A = \{(\Lambda_1, ..., \Lambda_n) \in \Gamma^n \mid d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta\}$ *is a convex subfamily of* Γ^n.

Proof. Suppose $(\Lambda_1, ..., \Lambda_n)$, $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in A$ and $\lambda \in [0, 1]$. Then $(\Lambda_1, ..., \Lambda_n)$, $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$, $d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta$, $d((\hat{\Omega}_1, ..., \hat{\Omega}_n), (\Omega_1, ..., \Omega_n)) < \delta$, and for each $i = 1, ..., n$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\hat{\Omega}_i, \Lambda_i, \lambda)$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in \Gamma^n$ for all $k \in N$.

Since

$$\begin{aligned}
&\lim_{k \to \infty} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) \\
&\quad = \lim_{k \to \infty} \left\{ \sum_{i=1}^n \|\chi_{V_i^k(\lambda)} - \chi_{\Omega_i}\|_{L_1}^2 \right\}^{1/2} \\
&\quad = \left(\sum_{i=1}^n \|\lambda \chi_{\Lambda_i} + (1-\lambda)\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L_1}^2 \right)^{1/2} \\
&\quad = \left(\sum_{i=1}^n [\lambda \|\chi_{\Lambda_i} - \chi_{\Omega_i}\|_{L_1} + (1-\lambda) \|\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L_1}]^2 \right)^{1/2} \\
&\quad \leqslant \lambda \left(\sum_{i=1}^n \|\chi_{\Lambda_i} - \chi_{\Omega_i}\|_{L_1}^2 \right)^{1/2} + (1-\lambda) \sum_{i=1}^n (\|\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L_1}^2)^{1/2} \\
&\quad = \lambda \left(\sum_{i=1}^n [\mu(\Lambda_i \Delta \Omega_i)^2] \right)^{1/2} + (1-\lambda) \left(\sum_{i=1}^n [\mu(\hat{\Omega}_i \Delta \Omega_i)^2]^{1/2} \right) \\
&\quad = \lambda d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) + (1-\lambda)\, d((\hat{\Omega}_1, ..., \hat{\Omega}_n), (\Omega_1, ..., \Omega_n)) \\
&\quad < \lambda\delta + (1-\lambda)\,\delta = \delta.
\end{aligned}$$

Hence there exists a natural number M such that

$$d((V_1^k(\lambda), \ldots, V_n^k(\lambda)), (\Omega_1, \ldots, \Omega_n)) < \delta \qquad \text{for} \quad k \geqslant M.$$

This shows that $(V_1^k(\lambda), \ldots, V_n^k(\lambda)) \in A$ for $k \geqslant M$ and that A is a convex subfamily of Γ^n.

DEFINITION 2.2. Let F: $\Gamma^n \to \mathbb{R}$ and $\mathscr{B} \subset \Gamma^n$. Then $(\Omega_1, \ldots, \Omega_n) \in \mathscr{B}$ is a global minimum of F on $\mathscr{B}$ if $F(\Omega_1, \ldots, \Omega_n) \leqslant F(\Lambda_1, \ldots, \Lambda_n)$ for all $(\Lambda_1, \ldots, \Lambda_n) \in \mathscr{B}$. $(\Omega_1, \ldots, \Omega_n)$ is a local minimum of F on $\mathscr{B}$ if there exists $\delta > 0$ such that $F(\Omega_1, \ldots, \Omega_n) \leqslant F(\Lambda_1, \ldots, \Lambda_n)$ for all $(\Lambda_1, \ldots, \Lambda_n) \in \mathscr{B}$ satisfying $d((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n)) < \delta$.

DEFINITION 2.3. A set function F: $\Gamma \to \mathbb{R}$ is differentiable at $\Omega \in \Gamma$ if there exists $f \in L_1(X, \Gamma, \mu)$, the derivative of F at Ω such that

$$F(\Lambda) = F(\Omega) + \langle f, \chi_\Lambda - \chi_\Omega \rangle + \mu(\Omega \, \Delta \Lambda) \, E(\Omega, \Lambda),$$

where $\lim_{\mu(\Omega \Delta \Lambda) \to 0} E(\Omega, \Lambda) = 0$.

DEFINITION 2.4. Let F: $\Gamma^n \to \mathbb{R}$ and $(\Omega_1, \ldots, \Omega_n) \in \Gamma^n$. Then F is said to have a partial derivative at $(\Omega_1, .., \Omega_n)$ with respect to Λ_i if the set function $H(\Lambda_i) = F(\Omega_1, \ldots, \Omega_{i-1}, \Lambda_i, \Omega_{i+1}, \ldots, \Omega_n)$ has derivative h_{Ω_i} at Ω_i. In this case we define the ith partial derivative of F at $(\Omega_1, \ldots, \Omega_n)$ to be $f^i_{\Omega_1, \ldots, \Omega_n} = h_{\Omega_i}$.

Now, we define the derivative of n-set functions.

DEFINITION 2.5. Let F: $S \to \mathbb{R}$ and $(\Omega_1, \ldots, \Omega_n) \in S$. Then F is said to be differentiable at $(\Omega_1, \ldots, \Omega_n) \in S$ if the partial $f^i_{\Omega_1, \ldots, \Omega_n}$, $i = 1, 2, \ldots, n$, exist and satisfy

$$\begin{aligned} F(\Lambda_1, \ldots, \Lambda_n) = F(\Omega_1, \ldots, \Omega_n) &+ \sum_{i=1}^{n} \langle f^i_{\Omega_1, \ldots, \Omega_n}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \\ &+ d((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n)) \\ &\times E((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n)), \qquad \text{for all} \quad (\Lambda_1, \ldots, \Lambda_n) \in S, \end{aligned}$$

where

$$\lim_{d((\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n)) \to 0} E[(\Omega_1, \ldots, \Omega_n), (\Lambda_1, \ldots, \Lambda_n)] = 0$$

and S is a nonempty subfamily of Γ^n.

Remark. (1) Definitions 2.4 and 2.5 are due to Corley [7].

(2) If F: $S \subset \Gamma^n \to \mathbb{R}$ is differentiable, its partial derivatives are unique [7].

(3) Throughout this paper, if F, G_j: $S \subset \Gamma^n \to \mathbb{R}$ are differentiable at $(\Omega_1, ..., \Omega_n) \in S$, then f^i_* and g^{ij}_* will denote the ith partial derivatives of F and G_j at $(\Omega_1, ..., \Omega_n)$, respectively.

3. Main Results

We can extend the concepts of quasiconvexity, strict quasiconvexity, and pseudoconvexity to set functions.

Definition 3.1. A set function F: $S \to \mathbb{R}$ is called quasiconvex (resp. convex) on a convex subfamily S of Γ^n if for each $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n)$ in S and $\lambda \in [0, 1]$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, 2, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqslant \max\{F(\Omega_1, ..., \Omega_n), F(\Lambda_1, ..., \Lambda_n)\}$$

$$(\text{resp. } \overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqslant \lambda F(\Lambda_1, ..., \Lambda_n) + (1-\lambda) F((\Omega_1, ..., \Omega_n))).$$

F is called quasiconcave on S if $-F$ is quasiconvex on S.

Definition 3.2. A set function F: $S \to \mathbb{R}$ is called strongly quasiconvex (resp. strictly quasiconvex) on a convex subfamily S of Γ^n if for each $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n)$ in S with $(\Omega_1, ..., \Omega_n) \neq (\Lambda_1, ..., \Lambda_n)$ (resp. $F(\Omega_1, ..., \Omega_n) \neq F(\Lambda_1, ..., \Lambda_n)$) and $\lambda \in (0, 1)$, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, 2, ..., n$ such that

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) < \max\{F(\Omega_1, ..., \Omega_n), F(\Lambda_1, ..., \Lambda_n)\}.$$

Remark. From Definition 3.1, it is easy to see that if F is a convex set function, then F is a quasiconvex set function, but the converse is not true; for example, if $g \in L_1(X, \Gamma, \mu)$ and S is a convex subfamily of Γ, let $G(\Omega) = (\int_\Omega g\, d\mu)^3$, $\Omega \in S$. It is easy to see from Proposition 3.1 that G is a quasiconvex set function, but G is not a convex set function.

Definition 3.3. Let S be a nonempty subfamily of Γ^n and let F: $S \to \mathbb{R}$

be differentiable on S. The set function F is said to be pseudoconvex on S if for each $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n)$ in S with

$$\sum_{i=1}^{n} \langle f_*^i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0$$

we have

$$F(\Lambda_1, ..., \Lambda_n) \geqslant F(\Omega_1, ..., \Omega_n).$$

The following proposition shows the existence of quasiconvex and pseudoconvex set functions.

PROPOSITION 3.1. *Let S be a convex subfamily of Γ^n and $F(\Lambda_1, ..., \Lambda_n) = u(\langle g_1, \chi_{\Lambda_1} \rangle, ..., \langle g_n, \chi_{\Lambda_n} \rangle)$, where u: $\mathbb{R}^n \to \mathbb{R}$ is a differentiable function, $g_1, ..., g_n \in L_1(X, \Gamma, \mu)$ and $(\Lambda_1, ..., \Lambda_n) \in S$.*

(a) *If u is a quasiconvex function, then F is a quasiconvex set function.*

(b) *If u is strictly quasiconvex, then F is a strictly quasiconvex set function.*

(c) *If u is a pseudoconvex function, then F is a pseudoconvex set function.*

Proof. (a) Assume that u is a quasiconvex function. Let $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n) \in S$ and $\lambda \in (0, 1)$. There exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Lambda_i, \Omega_i, \lambda)$ for each $i = 1, 2, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\begin{aligned}
&\overline{\lim_{k \to \infty}}\, F(V_1^k(\lambda), ..., V_n^k(\lambda)) \\
&\quad = \overline{\lim_{k \to \infty}}\, u(\langle g_1, \chi_{V_1^k(\lambda)} \rangle, ..., \langle g_n, \chi_{V_n^k(\lambda)} \rangle) \\
&\quad = u(\langle g_1, \lambda\chi_{\Omega_1} + (1-\lambda)\chi_{\Lambda_1} \rangle, ..., \langle g_n, \lambda\chi_{\Omega_n} + (1-\lambda)\chi_{\Lambda_n} \rangle) \\
&\quad = u[\lambda(\langle g_1, \chi_{\Omega_1} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle) + (1-\lambda)(\langle g_1, \chi_{\Lambda_1} \rangle, ..., \langle g_n, \chi_{\Lambda_n} \rangle)] \\
&\quad \leqslant \max\{u(\langle g_1, \chi_{\Omega_1} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle), u(\langle g_1, \chi_{\Lambda_1} \rangle, ..., \langle g_n, \chi_{\Lambda_n} \rangle)\} \\
&\quad = \max\{F(\Omega_1, ..., \Omega_n), F(\Lambda_1, ..., \Lambda_n)\}.
\end{aligned}$$

This shows that F is a quasiconvex set function.

(b) The proof of the strictly quasiconvex case is similar to (a).

(c) Suppose u is a pseudoconvex function. Let $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n) \in S$, then it follows from Definition 2.4 that

$$f_*^i = u_i(\langle g_1, \chi_{\Omega_1} \rangle, \langle g_2, \chi_{\Omega_2} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle)\, g_i,$$

where u_i denotes the ith partial derivative of u.

Hence if $\sum_{i=1}^{n} \langle f_*^i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0$, we have

$$\sum_{i=1}^{n} u_i(\langle g_1, \chi_{\Omega_1} \rangle, \langle g_2, \chi_{\Omega_2} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle) \langle g_i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0.$$

That is,

$$\nabla u(\langle g_1, \chi_{\Omega_1} \rangle, \langle g_2, \chi_{\Omega_2} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle)' \begin{bmatrix} \langle g_1, \chi_{\Lambda_1} \rangle - \langle g_1, \chi_{\Omega_1} \rangle \\ \vdots \\ \langle g_n, \chi_{\Lambda_n} \rangle - \langle g_n, \chi_{\Omega_n} \rangle \end{bmatrix} \geqslant 0.$$

Since u: $\mathbb{R}^n \to \mathbb{R}$ is a pseudoconvex function, if follows that

$$\begin{aligned} F(\Lambda_1, ..., \Lambda_n) &= u(\langle g_1, \chi_{\Lambda_1} \rangle, ..., \langle g_n, \chi_{\Lambda_n} \rangle) \\ &\geqslant u(\langle g_1, \chi_{\Omega_1} \rangle, ..., \langle g_n, \chi_{\Omega_n} \rangle) \\ &= F(\Omega_1, ..., \Omega_n). \end{aligned}$$

This shows that F is a pseudoconvex set function. Q.E.D.

PROPOSITION 3.2. *Let S be a convex subfamily of Γ^n and F: $S \to \mathbb{R}$ is a differentiable convex set function. Then F is a pseudoconvex set function.*

Proof. The proof of Proposition 3.2 follows immediately from the definition of pseudoconvex set functions and Theorem 4.5 of [7].

Remark. The converse of the above theorem is not true; for example, if $g \in L_1(X, \Gamma, \mu)$ and S is a convex subfamily of Γ, the set function F: $S \to \mathbb{R}$ is defined by $F(\Omega) = \int_\Omega g \, d\mu + (\int_\Omega g \, d\mu)^3$. It is easy to see that F is a pseudoconvex set function, but F is not a convex set function.

PROPOSITION 3.3 [15]. *Let S be a nonempty convex subfamily of Γ^n and let F: $S \to \mathbb{R}$ be differentiable and quasiconvex on S. If for any $(\Omega_1, ..., \Omega_n)$, $(\Lambda_1, ..., \Lambda_n) \in S$ with $F(\Lambda_1, ..., \Lambda_n) \leqslant F(\Omega_1, ..., \Omega_n)$ then*

$$\sum_{i=1}^{n} \langle f_*^i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \leqslant 0.$$

PROPOSITION 3.4. *Let S be a convex subfamily of Γ^n, and F: $S \to \mathbb{R}$. If for each real number α, the set $S_\alpha = \{(\Omega_1, ..., \Omega_n) \in S, F(\Omega_1, ..., \Omega_n) \leqslant \alpha\}$ is a convex subfamily of Γ^n, then F is a quasiconvex set function.*

Proof. Suppose that for each real number α, the set S_α is a convex subfamily of Γ^n. Let $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in S$ and $\lambda \in (0, 1)$. Note that $(\Omega_1, ..., \Omega_n)$ and $(\Lambda_1, ..., \Lambda_n) \in S_\alpha$ for $\alpha = \max\{F(\Omega_1, ..., \Omega_n), F(\Lambda_1, ..., \Lambda_n)\}$. By assumption, S_α is a convex subfamily of S, and there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \Lambda_i, \lambda)$ for each $i = 1, 2, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S_\alpha$ for all $k \in N$. Therefore

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqslant \alpha \qquad \text{for all} \quad k \in N.$$

Hence

$$\overline{\lim_{k \to \infty}} \; F(V_1^k(\lambda), ..., V_n^k(\lambda)) \leqslant \alpha = \max\{F(\Omega_1, ..., \Omega_n), F(\Lambda_1, ..., \Lambda_n)\}$$

and F is quasiconvex on S.

The following proposition relates a local optimal solution and a global optimal solution.

THEOREM 3.5. *Let S be a convex subfamily of Γ^n and let $F\colon S \to \mathbb{R}$ be strongly quasiconvex. Consider the problem to minimize $F(\Lambda_1, ..., \Lambda_n)$ subject to $(\Lambda_1, ..., \Lambda_n) \in S$. If $(\Omega_1, ..., \Omega_n)$ is a local optimal solution, then $(\Omega_1, ..., \Omega_n)$ is the unique global optimal solution.*

Proof. Since $(\Omega_1, ..., \Omega_n)$ is a local optimal solution, it follows that there exists a $\delta > 0$ such that

$$F(\Omega_1, ..., \Omega_n) \leqslant F(\Lambda_1, ..., \Lambda_n) \qquad \text{for} \quad (\Lambda_1, ..., \Lambda_n) \in S$$
$$\text{with} \quad d((\Lambda_1, ..., \Lambda_n), (\Omega_1, ..., \Omega_n)) < \delta. \quad (3)$$

Assume on the contrary that there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in S$ such that $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \neq (\Omega_1, ..., \Omega_n)$ and $F(\hat{\Omega}_1, ..., \hat{\Omega}_n) \leqslant F(\Omega_1, ..., \Omega_n)$. By the convexity of S and strong quasiconvexity of F, there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \hat{\Omega}_i, \lambda)$ for each $i = 1, 2, ..., n$ and $\lambda \in (0, 1)$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} \; F(V_1^k(\lambda), ..., V_n^k(\lambda)) < \max\{F(\hat{\Omega}_1, ..., \Omega_n), F(\Omega_1, ..., \Omega_n)\} = F(\Omega_1, ..., \Omega_n).$$

Since

$$\begin{aligned} d((V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) &= \left\{ \sum_{k=1}^{n} [\mu(V_i^k(\lambda) \, \Delta \Omega_i)]^2 \right\}^{1/2} \\ &= \left\{ \sum_{i=1}^{n} \|\chi_{V_i^k(\lambda)} - \chi_{\Omega_i}\|_{L_1}^2 \right\}^{1/2} \\ &\to \left\{ \sum_{i=1}^{n} \lambda^2 \, \|\chi_{\hat{\Omega}_i} - \chi_{\Omega_i}\|_{L_1}^2 \right\}^{1/2} \\ &= \lambda \left\{ \sum_{i=1}^{n} [\mu(\hat{\Omega}_i \, \Delta \Omega_i)]^2 \right\}^{1/2} \\ &= \lambda d((\hat{\Omega}_1, ..., \hat{\Omega}_n), (\Omega_1, ..., \Omega_n)). \end{aligned}$$

Hence there exists $\gamma > 0$ and a natural number M_1 such that

$$d(V_1^k(\lambda), ..., V_n^k(\lambda)), (\Omega_1, ..., \Omega_n)) < \delta \quad \text{for all} \quad 0 < \lambda < \gamma \text{ and } k \geqslant M_1.$$

Thus

$$F(\Omega_1, ..., \Omega_n) \leqslant F(V_1^k(\lambda), ..., V_n^k(\lambda)) \quad \text{for all} \quad 0 < \lambda < \gamma \text{ and } k \geqslant M_1. \tag{4}$$

Since

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) < F(\Omega_1, ..., \Omega_n),$$

it follows that there exists a natural number M_2 such that

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < F(\Omega_1, ..., \Omega_n) \quad \text{for} \quad k \geqslant M_2.$$

Let $M = \max\{M_1, M_2\}$, then

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < F(\Omega_1, ..., \Omega_n) \quad \text{for all } 0 < \lambda < \gamma \text{ and } k \geqslant M. \tag{5}$$

Inequality (4) is not compatible with (5). Therefore $(\Omega_1, ..., \Omega_n)$ is the unique global optimal solution. Q.E.D.

THEOREM 3.6. *Let S be a convex subfamily of Γ^n and let $F: S \to \mathbb{R}$ be a strictly quasiconvex set function. Consider the problem to minimize $F(A_1, ..., A_n)$ subject to $(A_1, ..., A_n) \in S$. If $(\Omega_1, ..., \Omega_n)$ is a local optimal solution, then $(\Omega_1, ..., \Omega_n)$ is also a global optimal solution.*

Proof. Assume on the contrary that there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in S$ such that $F(\hat{\Omega}_1, ..., \hat{\Omega}_n) < F(\Omega_1, ..., \Omega_n)$. Let $\lambda \in (0, 1)$, then there exists a Morris sequence $\{V_i^k(\lambda)\}$ in Γ associated with $(\Omega_i, \hat{\Omega}_i, \lambda)$ for each $i = 1, 2, ..., n$ such that $(V_1^k(\lambda), ..., V_n^k(\lambda)) \in S$ for all $k \in N$ and

$$\overline{\lim_{k \to \infty}} F(V_1^k(\lambda), ..., V_n^k(\lambda)) < \max\{F(\hat{\Omega}_1, ..., \hat{\Omega}_n), F(\Omega_1, ..., \Omega_n)\} = F(\Omega_1, ..., \Omega_n).$$

Hence there exists a natural number M_1 such that

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < F(\Omega_1, ..., \Omega_n) \quad \text{for} \quad k \geqslant M_1. \tag{6}$$

Since $(\Omega_1, ..., \Omega_n)$ is a local optimal solution, there exists a $\delta > 0$ such that

$$F(\Omega_1, ..., \Omega_n) \leqslant F(A_1, ..., A_n) \text{ for all } (A_1, ..., A_n) \in S \text{ with } d((\Omega_1, ..., \Omega_n), (A_1, ..., A_n)) < \delta. \tag{7}$$

As in the proof of Theorem 3.6, there exist $\gamma > 0$ and a natural number M_2 such that

$$d((\Omega_1, ..., \Omega_n), (V_1^k(\lambda), ..., V_n^k(\lambda))) < \delta \qquad \text{whenever} \quad 0 < \lambda < \gamma \text{ and } k \geqslant M_2.$$

Let $M = \max\{M_1, M_2\}$, then

$$d((\Omega_1, ..., \Omega_n), (V_1^k(\lambda), ..., V_n^k(\lambda))) < \delta$$

and

$$F(V_1^k(\lambda), ..., V_n^k(\lambda)) < F(\Omega_1, ..., \Omega_n) \qquad \text{for} \quad 0 < \lambda < \gamma \text{ and } k \geqslant M.$$

The above two inequalities lead to a contraction with (7). This shows that $(\Omega_1, ..., \Omega_n)$ is the global optimal solution. Q.E.D.

In [4, Corollary 3.6 Chou, Hsia, and Lee show that $\bar{\Gamma} = \{f \in L_\infty(X, \Gamma, \mu),$ $0 \leqslant f \leqslant 1\}$, where $\bar{\Gamma}$ denotes the weak*-closure of Γ.

DEFINITION 3.4. Let A be a nonempty subfamily of Γ and let $g = (g_1, ..., g_n) \in (\bar{A})^n = \{h \mid h = (h_1, ..., h_n),\ \ h_i \in \bar{A},\ \ i = 1, ..., n\}$, where $\bar{A}$ denotes the weak*-closure of A. The cone of tangents of $(\bar{A})^n$ at g denoted by T is the set $\{h \mid h = (h_1, ..., h_n) \in L_\infty \times \cdots \times L_\infty$ and $\lambda_k(\chi_{\Omega_i^k} - g_i) \xrightarrow{w^*} h_i$, where $\lambda_k > 0$, $\Omega_i^k \in A$, and $\chi_{\Omega_i^k} \xrightarrow{w^*} g_i\}$.

The following theorem gives a necessary condition for the existence of an optimal solution.

THEOREM 3.7. *Let A^n be a nonempty subfamily of Γ^n and let $\bar{A}$ denote the weak*-closure of A in $L_\infty(X, \Gamma, \mu)$. Let $(\Omega_1, ..., \Omega_n) \in A^n$. Suppose $F\colon A \to \mathbb{R}$ is differentiable at $(\Omega_1, ..., \Omega_n)$ and $(\Omega_1, ..., \Omega_n)$ locally solves the problem to minimize $F(A_1, ..., A_n)$ subject to $(A_1, ..., A_n) \in A$. Then $F_0 \cap T = \varnothing$, where $F_0 = \{g = (g_1, ..., g_n) \in L_\infty^n(X, \Gamma, \mu) \mid \sum_{i=1}^n \langle f_*^i, g_i\rangle < 0\}$, $\langle\ ,\ \rangle$ denotes the dual pair between $L_\infty(X, \Gamma, \mu)$ and $L_1(X, \Gamma, \mu)$, and T is the cone of tangents of $(\bar{A})^n$ at $(\chi_{\Omega_1}, ..., \chi_{\Omega_n})$.*

Proof. Let $(g_1, ..., g_n) \in T$. Then there exists $\lambda_k > 0$, $\Omega_i^k \in A$ for each $k \in N$ and for each $i = 1, 2, ..., n$ such that $\chi_{\Omega_i^k} \xrightarrow{w^*} \chi_{\Omega_i}$ and $\lambda_k(\chi_{\Omega_i^k} - \chi_{\Omega_i}) \xrightarrow{w^*} g_i$. By the differentiability of F at $(\Omega_1, ..., \Omega_n)$, we get

$$F(\Omega_1^k, ..., \Omega_n^k) = F(\Omega_1, ..., \Omega_n) + \sum_{i=1}^n \langle f_*^i, \chi_{\Omega_i^k} - \chi_{\Omega_i}\rangle + \left(\sum_{i=1}^n \|\chi_{\Omega_i^k} - \chi_{\Omega_i}\|_{L_1}^2\right)^{1/2} E((\Omega_1, ..., \Omega_n), (\Omega_1^k, ..., \Omega_n^k)), \qquad (8)$$

where $E((\Omega_1, ..., \Omega_n), (\Omega_1^k, ..., \Omega_n^k)) \to 0$ as $d((\Omega_1, ..., \Omega_n), (\Omega_1^k, ..., \Omega_n^k)) \to 0$.

Since $(\Omega_1, ..., \Omega_n)$ is the local optimal solution, it follows that there exists a $\delta > 0$ such that

$$F(\Omega_1, ..., \Omega_n) \leqslant F(\Lambda_1, ..., \Lambda_n) \quad \text{whenever} \quad d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) < \delta. \tag{9}$$

Since for each $i = 1, 2, ..., n$, $\chi_{\Omega_i^k} \xrightarrow{w^*} \chi_{\Omega_i}$, it follows that there exists $M > 0$ such that

$$d((\Omega_1^k, ..., \Omega_n^k), (\Omega_1, ..., \Omega_n)) = \left(\sum_{i=1}^{n} \|\chi_{\Omega_i^k} - \chi_{\Omega_i}\|_{L_1}^2 \right)^{1/2} < \delta \quad \text{whenever } k \geqslant M.$$

By (8) and (9), we get

$$\sum_{i=1}^{n} \langle f_*^i, \chi_{\Omega_i^k} - \chi_{\Omega_i} \rangle + \left(\sum_{i=1}^{n} \|\chi_{\Omega_i^k} - \chi_{\Omega_i}\|_{L_1}^2 \right)^{1/2} \times E[(\Omega_1, ..., \Omega_n), (\Omega_1^k, ..., \Omega_n^k)] \geqslant 0 \quad \text{whenever} \quad k \geqslant M.$$

Multiplying by λ_k and taking the limit as $k \to \infty$, we obtain

$$\sum_{i=1}^{n} \langle f_*^i, g_i \rangle \geqslant 0.$$

So far we have shown that $g \in T$ implies that

$$\sum_{i=1}^{n} \langle f_*^i, g_i \rangle \geqslant 0,$$

and $F_0 \cap T = \varnothing$. The proof is complete. Q.E.D.

The following theoem generalizes Theorem 4.7 of [7] and gives sufficient conditions for the existence of optimal solutions to problem (P) with equality constraints.

THEOREM 3.8. *Let S be a nonempty convex subfamily of Γ^n, $(\Omega_1, ..., \Omega_n)$ a feasible solution to problem* (P), *and $I = \{i \mid G_i(\Omega_1, ..., \Omega_n) = 0\}$. Suppose that F, G_j for $j \in I$ and H_j for $j = 1, 2, ..., l$ are differentiable on S and that the Kuhn–Tucker condition holds at $(\Omega_1, ..., \Omega_n)$; that is, there exist scalars $u_i \geqslant 0$ for $i \in I$ and v_i for $i = 1, 2, ..., l$ such that*

$$\sum_{i=1}^{n} \langle f_*^i, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j \in I} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{l} \sum_{i=1}^{n} v_j \langle h_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0, \tag{10}$$

where h_*^{ij} denotes the ith *partial derivative of H_j at $(\Omega_1, ..., \Omega_n)$. Let $J = \{i: v_i > 0\}$ and $K = \{i: v_i < 0\}$. Further suppose that F is pseudoconvex on S and G_i is quasiconvex on S for $i \in I$, H_i is quasiconvex on S for $i \in J$, and H_i is quasiconcave on S for $i \in K$. Then $(\Omega_1, ..., \Omega_n)$ is a global optimal solution.*

Proof. Let $(\Lambda_1, ..., \Lambda_n)$ be a feasible solution to problem (P). Then $G_i(\Lambda_1, ..., \Lambda_n) \leqslant G_i(\Omega_1, ..., \Omega_n)$ for $i \in I$. In view of Proposition 3.3, we have

$$\sum_{i=1}^{n} \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \leqslant 0 \qquad \text{for} \quad j \in I. \tag{11}$$

Similarly, we have

$$\sum_{i=1}^{n} \langle h_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \leqslant 0 \qquad \text{for} \quad j \in J \tag{12}$$

$$\sum_{i=1}^{n} \langle h_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0 \qquad \text{for} \quad j \in K. \tag{13}$$

Multiplying (11), (12), and (13) respectively by $u_j \geqslant 0$, $v_j > 0$, and $v_j < 0$ and adding, we get

$$\sum_{j \in I} \sum_{i=1}^{n} u_j \langle g_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j \in J \cup K} \sum_{i=1}^{n} v_j \langle h_*^{ij}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \leqslant 0.$$

It follows from (10), we have

$$\sum_{i=1}^{n} \langle f_*^{i}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0.$$

By pseudoconvexivity of F, we have $F(\Lambda_1, ..., \Lambda_n) \geqslant F(\Omega_1, ..., \Omega_n)$ and the proof is complete. Q.E.D.

Remark. In Theorem 4.7 of [7], the problem (P) does not have the equality constraint, and the functions F, G_i, $i = 1, 2, ..., m$, are assumed to be convex. In Theorem 3.8 if we let $H_i = 0$, $i = 1, 2, ..., l$, and assume that F, G_i, $i = 1, 2, ..., m$, are convex, then in view of Proposition 3.2, Theorem 3.8 reduces to Theorem 4.7 of [7].

DEFINITION 3.5. A differentiable set function $F: \Gamma^n \to \mathbb{R}$ is said to be locally convex at $(\Omega_1, ..., \Omega_n) \in \Gamma^n$ if there exists $\delta > 0$ such that

$$F(\Lambda_1, ..., \Lambda_n) \geqslant F(\Omega_1, ..., \Omega_n) + \sum_{i=1}^{n} \langle f_*^{i}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle$$
$$\text{for all } (\Lambda_1, ..., \Lambda_n) \in \Gamma^n \text{ with } d((\Omega_1, ..., \Omega_n), (\Lambda_1, ..., \Lambda_n)) \leqslant \delta.$$

Remark. It follows from Theorem 4.5 of [7] that if $F: \Gamma^n \to \mathbb{R}$ is differentiable and convex on Γ^n, then Γ is locally convex.

4. Duality Theorem for Set Functions

In this section let $F: \Gamma^n \to \mathbb{R}$ and $G_i: \Gamma^n \to \mathbb{R}$, $i = 1, 2, ..., m$, be differentiable set functions. We consider the following problem:

$$\begin{aligned} &\text{minimize } F(\Lambda_1, ..., \Lambda_n) \\ &\text{subject to } (\Lambda_1, ..., \Lambda_n) \in \Gamma^n,\ G_i(\Lambda_1, ..., \Lambda_n) \leqslant 0 \end{aligned} \qquad (\mathrm{P}')$$

for $i = 1, 2, ..., m$.

Then we formulate the dual problem of (P′) by

$$\begin{aligned} &\text{maximize } F(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{i=1}^{m} u_i G_i(\bar{\Omega}_1, ..., \bar{\Omega}_n) \\ &\text{subject to } u_i \geqslant 0,\ i = 1, 2, ..., m,\ (\bar{\Omega}_1, ..., \bar{\Omega}_n) \in \Gamma^n, \end{aligned} \qquad (\mathrm{Q})$$

and

$$\sum_{i=1}^{n} \langle f^i_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle + \sum_{j=1}^{m} \sum_{i=1}^{n} u_j \langle g^{ij}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle \geqslant 0,$$

for all $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$, where f^i_{**} and g^{ij}_{**} denote the ith partial derivative of F and G_j at $(\bar{\Omega}_1, ..., \bar{\Omega}_n)$, respectively.

Lemma 4.1 [7, Corollary 3.9]. *Let $F, G_1, ..., G_m: \Gamma^n \to \mathbb{R}$ be differentiable at $(\Omega_1, ..., \Omega_n)$. If $(\Omega_1, ..., \Omega_n)$ is a local minimum for* (P′) *and if there exists $(\bar{\Omega}_1, ..., \bar{\Omega}_n) \in \Gamma^n$ for which*

$$G_j(\Omega_1, ..., \Omega_n) + \sum_{i=1}^{n} \langle g^{ij}_*,\ \chi_{\bar{\Omega}_i} - \chi_{\Omega_i} \rangle < 0,$$

then there exist scalars $\lambda_1, ..., \lambda_m$ such that

$$\left\langle f^i_* + \sum_{j=1}^{m} \lambda_j g^{ij}_*, \chi_{\Lambda_i} - \chi_{\Omega_i} \right\rangle \geqslant 0 \qquad \textit{for all} \quad \Lambda_i \in \Gamma,\ i = 1, 2, ..., n,$$

$$\lambda_j G_j(\Omega_1, ..., \Omega_n) = 0, \qquad j = 1, 2, ..., m,\ \lambda_1, ..., \lambda_m \geqslant 0$$

$$G_j(\Omega_1, ..., \Omega_n) \leqslant 0, \qquad j = 1, 2, ..., m.$$

We say $(\Omega_1, ..., \Omega_n, u_1, ..., u_m)$ solves problem (Q) locally if $(\Omega_1, ..., \Omega_n, u_1, ..., u_m)$ is a feasible solution to (Q) and there exist $\delta > 0$ such that $F(\Omega_1, ..., \Omega_n) + \sum_{i=1}^{m} u_i G_i(\Omega_1, ..., \Omega_n) \geqslant F(\bar{\Lambda}_1, ..., \bar{\Lambda}_n) + \sum_{i=1}^{m} \bar{u}_i G_i(\bar{\Lambda}_1, ..., \bar{\Lambda}_n)$, for any feasible solution $(\bar{\Lambda}_1, ..., \bar{\Lambda}_n, \bar{u}_i, ..., \bar{u}_m)$ to (Q) with $d((\Omega_1, ..., \Omega_n), (\bar{\Lambda}_1, ..., \bar{\Lambda}_n)) < \delta$.

THEOREM 4.2. *Suppose that F and G_j, $j = 1, 2, ..., m$, are locally convex on Γ^n. If $(\Omega_1, ..., \Omega_n)$ is a local minimum for problem (P′) and if there exists $(\hat{\Omega}_1, ..., \hat{\Omega}_n) \in \Gamma^n$ for which*

$$G_j(\Omega_1, ..., \Omega_n) + \sum_{i=1}^{n} \langle g^{ij}_{*}, \chi_{\hat{\Omega}_i} - \chi_{\Omega_i} \rangle < 0, \qquad j = 1, 2, ..., m,$$

then there exists $(\hat{u}_1, ..., \hat{u}_m) \geqslant 0$ such that $(\Omega_1, ..., \Omega_n, \hat{u}_1, ..., \hat{u}_m)$ solves the problem (Q) locally. Furthermore, the local minimum of (P′) at $(\Omega_1, ..., \Omega_n)$ is equal to the local maximum of (Q) at $(\Omega_1, ..., \Omega_n, \hat{u}_1, ..., \hat{u}_m)$.

Proof. Let $(\bar{\Omega}_1, ..., \bar{\Omega}_n, \bar{u}_1, ..., \bar{u}_m)$ be a feasible solution to (Q). Then $\bar{u} = (\bar{u}_1, ..., \bar{u}_m) \geqslant 0$ and

$$\sum_{i=1}^{n} \langle f^{i}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle + \sum_{j=1}^{m} \sum_{i=1}^{n} \bar{u}_j \langle g^{ij}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle \geqslant 0$$

for all $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$.

Since F and G_j, $j = 1, 2, ..., m$, are locally convex, there exists $\delta > 0$ such that $d((\Lambda_1, ..., \Lambda_n), (\bar{\Omega}_1, ..., \bar{\Omega}_n)) < \delta$ implies

$$F(\Lambda_1, ..., \Lambda_n) \geqslant F(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{i=1}^{n} \langle f^{i}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle$$

and

$$G_j(\Lambda_1, ..., \Lambda_n) \geqslant G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{i=1}^{n} \langle g^{ij}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle, \qquad j = 1, 2, ..., m.$$

Now for $d((\Lambda_1, ..., \Lambda_n), (\bar{\Omega}_1, ..., \bar{\Omega}_n)) < \delta$

$$\begin{aligned}
&F(\Lambda_1, ..., \Lambda_n) - \left[F(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{j=1}^{m} \bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n) \right] \\
&\quad \geqslant \sum_{i=1}^{n} \langle f^{i}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle - \sum_{j=1}^{m} \bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n) \\
&\quad \geqslant - \sum_{j=1}^{m} \sum_{i=1}^{n} \bar{u}_j \langle g^{ij}_{**}, \chi_{\Lambda_i} - \chi_{\bar{\Omega}_i} \rangle - \sum_{j=1}^{m} \bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n) \\
&\quad \geqslant - \sum_{j=1}^{m} \bar{u}_j [G_j(\Lambda_1, ..., \Lambda_n) - G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n)] - \sum_{j=1}^{m} \bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n) \\
&\quad = - \sum_{j=1}^{m} \bar{u}_j G_j(\Lambda_1, ..., \Lambda_n) \geqslant 0.
\end{aligned}$$

Thus

$$F(\Lambda_1, ..., \Lambda_n) \geqslant F(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{j=1}^{m} [\bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n)]$$

for any feasible solution $(\bar{\Omega}_1, ..., \bar{\Omega}_n, \bar{u}_1, ..., \bar{u}_n)$ to problem (Q) and any $(\Lambda_1, ..., \Lambda_n) \in \Gamma^n$ with $d((\Lambda_1, ..., \Lambda_n), (\bar{\Omega}_1, ..., \bar{\Omega}_n)) < \delta$.

As $(\Omega_1, ..., \Omega_n)$ is a local optimal solution to problem (P′), it follows from Lemma 4.1 that there exists $\hat{u} = (\hat{u}_1, ..., \hat{u}_m) \geqslant 0$ such that

$$\hat{u}_j G_j(\Omega_1, ..., \Omega_n) = 0, \qquad j = 1, 2, ..., m. \tag{14}$$

and

$$\sum_{i=1}^{n} \langle f^i_{**}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle + \sum_{j=1}^{m} \sum_{i=1}^{n} \hat{u}_j \langle g^{ij}_{**}, \chi_{\Lambda_i} - \chi_{\Omega_i} \rangle \geqslant 0 \qquad \text{for all} \quad (\Lambda_1, ..., \Lambda_n) \in \Gamma^n. \tag{15}$$

In other words, $(\Omega_1, ..., \Omega_n, \hat{u}_1, ..., \hat{u}_m)$ is a feasible solution to (Q). By (14) and (15),

$$F(\Omega_1, ..., \Omega_n) + \sum_{j=1}^{m} \hat{u}_j G_j(\Omega_1, ..., \Omega_n)$$
$$= F(\Omega_1, ..., \Omega_n) \geqslant F(\bar{\Omega}_1, ..., \bar{\Omega}_n) + \sum_{j=1}^{m} \bar{u}_j G_j(\bar{\Omega}_1, ..., \bar{\Omega}_n)$$

holds for any feasible solution $(\bar{\Omega}_1, ..., \bar{\Omega}_n, \bar{u}_1, ..., \bar{u}_m)$ to problem (Q) with $d((\Omega_1, ..., \Omega_n), (\bar{\Omega}_1, ..., \bar{\Omega}_n)) < \delta$. This shows that $(\Omega_1, ..., \Omega_n, \hat{u}_1, ..., \hat{u}_m)$ solves problem (Q) locally and the locally minimum value of (P′) at $(\Omega_1, ..., \Omega_n)$ is equal to the local maximum value of (Q) at $(\Omega_1, ..., \Omega_n, \hat{u}_1, ..., \hat{u}_m)$. Q.E.D.

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