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Optimization of Set-Valued Functions*

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Let X, Y, and Z be real topological vector spaces and $E \subseteq X$ be a convex set. $C \subseteq Y$, $D \subseteq Z$ are to be pointed convex cones. Let F: $X \to 2^Y$ be C-convex and G: $X \to 2^Z$ be D-convex set-valued functions.

We consider the problems

$$V - \underset{x \in E}{\text{minimize}}\ F(x), \text{ subject to } x \in G^{-}(-D). \qquad \text{(P)}$$

This paper generalizes the Moreau–Rockafellar type theorem and the Farkas–Minkowski type theorem for set-valued functions. When $Y = \mathbb{R}^n$ and $Z = \mathbb{R}^m$, we established the necessary and sufficient conditions for the existence of Geoffrion efficient solution of (P) and the relationship between the proper efficient solutions and Geoffrion efficient solutions of (P). The Mond–Weir type and Wolfe type vector duality theorems are also considered in this paper. © 1994 Academic Press, Inc.

1. INTRODUCTION

Let X, Y, and Z be real topological vector spaces and $C \subseteq Y$, $D \subseteq Z$ be pointed convex cones. Let F: $X \to 2^Y$ be C-convex and G: $X \to 2^Z$ be D-convex set-valued functions. We consider the problem

$$V - \underset{x \in E}{\text{minimize}}\ F(x), \text{ subject to } x \in G^{-}(-D), \qquad \text{(P)}$$

i.e., to find all $x_0 \in E \cap G^{-}(-D)$ for which $y_0 \in F(x_0)$ and $y_0 \in w - \min F[E \cap G^{-}(-D)]$ or y_0 is a Geoffrion efficient value of (P).

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This type of problem has a wide range of applications. For example, Klein and Thompson [9] surveyed their use in economics, in addition to presenting their theory, Zangwill [16] used them to present a unified treatment of convergence of nonlinear programming algorithms, while Hogan [7] studied their properties from this viewpoint. Generalized equations [10] and differential inclusions [4] are other applications.

In this paper, we define the weak-subdifferential of set-valued functions, and generalize the Moreau–Rockafellar type theorem for set-valued functions. We prove that if $z_1 \in F_1(u)$, $z_2 \in F_2(u)$, then

$$\partial_w(F_1 + F_2)(u; z_1 + z_2) \subseteq \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2) \tag{1}$$

holds, but not reverse inclusion; a counterexample [15] shows that equality in (1) does not hold. If F_1, F_2 are real single-valued functions, then this theorem reduces to the Moreau–Rockafellar theorem.

We also generalize the Farkas–Minkowski type theorem for set-valued functions. By applying the Moreau–Rockafellar type theorem and the Farkas–Minkowski type theorem and other results for set-valued functions, we establish the Kuhn–Tucker necessary conditions for the existence of a weak minimum of the problem (P).

When $Y = \mathbb{R}^n$ and $Z = \mathbb{R}^m$, we also established the relationship between the proper efficient solutions and Geoffrion efficient solution of (P) and the necessary and sufficient conditions for the existence of Geoffrion efficient solutions of (P).

In the final section, the duality theorems and weak duality theorems of Mond–Weir and Wolfe types of vector set-valued functions are also established.

Because the objective functions of duality problems are single-valued functions, we deduce the optimization problem of set-valued functions to the optimization problem of the usual single-valued functions. Hence this paper provides a useful method for solving the optimization problem of set-valued functions.

2. Preliminary

Throughout this paper let X, Y, and Z be real topological vector spaces, each with zero element θ, and let $F: X \to 2^Y$ and $G: X \to 2^Z$ be set-valued functions. The domain of F is given by $D(F) = \{x \in X: F(x) \neq \varnothing\}$.

A set C in Y is a cone if $\lambda y \in C$, for all $y \in C$ and $\lambda \geq 0$.

A convex cone is one for which $\lambda_1 y_1 + \lambda_2 y_2 \in C$ for all $\lambda_1, \lambda_2 \geq 0$, y_1, $y_2 \in C$.

A pointed cone is one for which $C \cap (-C) = \{\theta\}$.

Let X^*, Y^*, and Z^* be the dual spaces of X, Y, and Z, respectively, and $\langle \cdot, \cdot \rangle$ be the dual pairs. Let $C \subset Y$ and $D \subset Z$ be the pointed convex cones. The polar cone C^* of C is $C^* = \{y^* \in Y^* \mid \langle y^*, y \rangle \geq 0 \text{ for all } y \in C\}$. If E, $F \subseteq Y$, $\alpha \in \mathbb{R}$, we define $E + F = \{x + y \mid x \in E, y \in F\}$, $\alpha E = \{\alpha x : x \in E\}$, and int E as the set of interior point of E.

In adding and multiplying sets by scalars, the convention is made $A + \emptyset = \emptyset$ and $\alpha\emptyset = \emptyset$ for any α. Let $B(Z, Y)$ be the set of all continuous linear operators from Z to Y and $B^+(Z, Y) = \{w \in B(Z, Y) \mid w(D) \subset C\}$. For $y_1, y_2 \in Y$, we write

$$y_1 \leq_C y_2 \quad \text{if } y_2 - y_1 \in C \backslash \{\theta\},$$

$$y_1 \leqq_C y_2 \quad \text{if } y_2 - y_1 \in C,$$

and

$$y_1 <_C y_2 \quad \text{if } y_2 - y_1 \in \text{int } C.$$

A point $y_0 \in B \subset Y$ is called a weak minimal element of B, denoted by $y_0 \in w - \min B$, if there does not exist $y \in B$ such that $y <_C y_0$.

Throughout this paper, let Y be ordered by the pointed convex cone $C \subseteq Y$ and Z be ordered by the pointed convex cone $D \subset Z$, and let C and D have non-empty interior. For $A \subseteq X$, $F: X \to 2^Y$, $G: X \to 2^Z$, $V \subseteq Z$, we denote

$$F(A) = \bigcup_{x \in A} F(x) \quad \text{and} \quad G^-(V) = \{x \in X: G(x) \cap V \neq \emptyset\}.$$

For $y_0 \in Y$, we denote $F(x) <_C y_0$, $F(x) \leq_C y_0$, and $F(x) \leqq_C y_0$ if for all $y \in F(x_0)$, $y <_C y_0$, $y \leq_C y_0$, and $y \leqq_C y_0$, respectively.

The zero vector in $\mathbb{R}^n$ is also denoted by 0. The nonnegative and nonpositive orthants are denoted by $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, respectively.

Let $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{R}^n$. We write $a < b$ if $a <_{\mathbb{R}^n_+} b$, $a \leq b$ if $a \leq_{\mathbb{R}^n_+} b$, and $a \leqq b$ if $a \leqq_{\mathbb{R}^n_+} b$.

DEFINITION 1[2]. Let $A \subset X$ be convex and let $F: X \to 2^Y$. Then F is C-convex on A if for any $x_1, x_2 \in A$, $\lambda \in [0,1]$

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F(\lambda x_1 + (1 - \lambda)x_2) + C.$$

DEFINITION 2. Let $A \subset X$ be convex and let $F: X \to 2^Y$; then F is strictly C-convex on A if, for any $x_1, x_2 \in A$, $x_1 \neq x_2$, $\lambda \in (0,1)$,

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F[\lambda x_1 + (1 - \lambda)x_2] + \text{int } C.$$

DEFINITION 3[14]. Let $F: A \subseteq X \to 2^Y$, $u \in A$ and $z \in F(u)$. A linear operator $T \in B(X, Y)$ is said to be a weak subgradient for z of F at u if

$$z - T(u) \in w - \min \bigcup_{v \in A} \{F(v) - T(v)\}.$$

The set of all weak subgradients for z of F at u is called the weak subdifferential for z at u and is denoted by $\partial_w F(u; z)$. Moreover, F is said to be weakly subdifferentiable at u if for all $z \in F(u)$, $\partial_w F(u; z) \neq \emptyset$.

Remark. If $\bar{x} \in A$ and $y_0 \in F(\bar{x})$, it is easy to see from Definition 2 that $y_0 \in w - \min \bigcup_{v \in A} F(v)$ if and only if $0 \in \partial_w F(\bar{x}; y_0)$.

DEFINITION 4[14]. The set-valued function $F: A \subseteq X \to 2^Y$ is said to be connected at $u \in A$ if there exists a continuous function $H: A \to Y$ such that $H(v) \in F(v)$ for all v in some neighborhood of u.

DEFINITION 5. Given a set A in $\mathbb{R}^p$, a point $\bar{a} \in \mathbb{R}^p$ is said to be a lower efficient (resp. upper efficient) point of A if $\bar{a} \in A$ and there is no $a' \in A$ such that $a' \leq \bar{a}$ (resp. $a' \geq \bar{a}$). We denote this by $\bar{a} \in \underline{\text{eff}}\, A$ (resp. $\bar{a} \in \overline{\text{eff}}\, A$).

DEFINITION 6[14]. In problem (P), if $F: E \to 2^{\mathbb{R}^n}$, $G: E \to 2^{\mathbb{R}^m}$, a point $x_0 \in E \cap G^-(\mathbb{R}^m_-)$ is called a proper efficient solution to problem (P) if there exists y such that $y \in F(x_0)$ and

$$\overline{F[E \cap G^-(\mathbb{R}^m_-)] + \mathbb{R}^n_+ - y} \cap (\mathbb{R}^m_-) = \{0\}.$$

For two sets A and B in $\mathbb{R}$, we denote $A \leq B$ (resp. $A < B$) if $x \leq y$ (resp. $x < y$) for all $x \in A$, $y \in B$.

DEFINITION 7. In problem (P), if $F = (F_1, \ldots, F_n): E \to 2^{\mathbb{R}^n}$, $G: E \to 2^{\mathbb{R}^m}$, a point $x_0 \in E \cap G^-(\mathbb{R}^m_-)$ is a Geoffrion efficient solution to the problem (P) if there exists $y = (y_1, \ldots, y_n) \in F(x_0)$ such that $y \in \underline{\text{eff}}\, F[E \cap G^-(\mathbb{R}^m_-)]$ and if there exists $M > 0$ such that for each i and $x \in E \cap G^-(\mathbb{R}^m_-)$, $w_i \in F_i(x)$ satisfying $w_i < y_i$, there exist $1 \leq j \leq n$, and $w_j \in F_j(x)$ with $w_j > y_j$ and $y_i - w_i \leq M(w_j - y_j)$.

In this case y is called the Geoffrion efficient value of the problem (P).

Let $H: E \to 2^{\mathbb{R}}$; we call $x_0 \in E \cap G^-(\mathbb{R}_-)$ a minimal solution of the problem.

Minimize $H(x)$, subject to $x \in E$, $G(x) \cap \mathbb{R}_- \neq \emptyset$, if there exists $y \in H(x_0)$ such that $y = \min H\{E \cap G^-(\mathbb{R}_-)\}$. (P′)

3. MAIN RESULTS

In Theorem 3.1, we prove the Moreau–Rockafellar type theorem for set-valued functions.

THEOREM 3.1. *Let F_1 and F_2 be set-valued functions from the set $E = \{v \in X \mid F_1(v) \neq \emptyset, F_2(v) \neq \emptyset\}$ into 2^Y, E be convex, and F_1 and F_2 be C-convex on E. If F_1 is connected at some $u_0 \in \operatorname{int} E$, then for $u \in E$ and $z_1 \in F_1(u)$, $z_2 \in F_2(u)$, we have*

$$\partial_{w'}(F_1 + F_2)(u; z_1 + z_2) \subseteq \partial_{w'}F_1(u; z_1) + \partial_{w'}F_2(u; z_2).$$

Proof. Let $T \in \partial_{w'}(F_1 + F_2)(u; z_1 + z_2)$ and define

$$H_1(v) = F_1(v) - z_1 - T(v - u), \qquad H_2(v) = F_2(v) - z_2.$$

Since $F_1, F_2\colon E \to 2^Y$ are C-convex, it follows that H_1 and H_2 are C-convex set-valued functions and $\theta \in H_1(u) \cap H_2(u)$.

Because $T \in \partial_{w'}(F_1 + F_2)(u; z_1 + z_2)$, it follows that

$$z_1 + z_2 - T(u) \in w - \min \bigcup_{v \in E} [(F_1 + F_2)(v) - T(v)].$$

This implies that

$$\theta \in w - \min \bigcup_{v \in E} (F_1 + F_2)(v) - (z_1 + z_2) - T(v - u).$$

But $(H_1 + H_2)(v) = (F_1 + F_2)(v) - (z_1 + z_2) - T(v - u)$.

This shows that $0 \in \partial_{w'}(H_1 + H_2)(u; \theta)$ and $\theta \in w - \min \bigcup_{v \in E} (H_1 + H_2)(v)$. We define

$$A = \operatorname{epi} H_1 = \{(u, z) \in E \times Y \mid z \in H_1(u) + C\},$$
$$B = \{(u, -z) \in E \times Y \mid (u, z) \in \operatorname{epi} H_2\}.$$

Since H_1 and H_2 are C-convex, it follows that A and B are convex subsets of $E \times Y$. Because F_1 is connected at $u_0 \in \operatorname{int} E$, H_1 is connected at u_0 and A has a nonempty interior.

We wish to show that $(\operatorname{int} A) \cap B = \emptyset$. Suppose that $(v, y) \in (\operatorname{int} A) \cap B$; then there exist $a \in \operatorname{int} C$, $w_1 \in H_1(v)$, $w_2 \in H_2(v)$, such that,

$$w_1 <_C y - a \quad \text{and} \quad w_2 \leqq_C -y.$$

Thus $w_1 + w_2 <_C -a <_C \theta$.

This contradicts $\theta \in w - \min \bigcup_{v \in E} (H_1 + H_2)(v)$ and shows that $(\operatorname{int} A) \cap B = \emptyset$.

Hence there exist nonzero $(w^*, z^*) \in X^* \times Y^*$ and $\alpha \in \mathbb{R}$ such that

$$\langle w^*, v\rangle + \langle z^*, a\rangle \leq \alpha \leq \langle w^*, w\rangle + \langle z^*, b\rangle \tag{2}$$

for all $(v, a) \in A$ and $(w, b) \in B$. Furthermore $\langle w^*, v\rangle + \langle z^*, a\rangle < \alpha$ for all $(v, a) \in \text{int } A$.

Because $(u, \theta) \in A \cap B$ it follows that $\alpha = \langle w^*, u\rangle$. Next we wish to show that $z^* \neq 0$. Suppose that $z^* = 0$; then $\langle w^*, u_0 - w\rangle < 0$ for all $w \in E$. Since $u_0 \in \text{int } E$, this leads to a contradiction. Hence $z^* \neq 0$.

On the other hand if there exists $\bar{y} \in C$ such that $\langle z^*, \bar{y}\rangle > 0$, then by taking any $v \in E$ and sufficiently large $\lambda > 0$, we have

$$\langle w^*, v\rangle + \langle z^*, z + \lambda\bar{y}\rangle > \alpha \qquad \text{for any } z \in H_1(v).$$

This contradicts (2). Hence $z^* \in -C^*$ is nonzero. Let $\bar{z} \in -\text{int } C \neq \emptyset$ satisfy $\langle z^*, \bar{z}\rangle = 1$.

We define $T_1: X \to Y$ by $T_1(v) = -\langle w^*, v\rangle\, \bar{z}$.

We wish to show that

$$T_1 \in \partial_w H_1(u; \theta) \cap (-\partial_w H_2(u; \theta)).$$

For if $T_1 \notin \partial_w H_1(u; \theta)$, then there exist $z' \in Y$ and $u' \in E$, such that $z' \in H_1(u')$ and $z' - T_1(u') <_C \theta - T_1(u)$.

Hence $z' + \langle w^*, u'\rangle\bar{z} <_C \theta + \langle w^*, u\rangle\bar{z}$.

Since $z^* \in -C^*$ is nonzero, it follows that $\langle z^*, y\rangle < 0$ for all $y \in \text{int } C$. Therefore

$$\langle z^*, z'\rangle + \langle w^*, u'\rangle > \langle w^*, u\rangle = \alpha.$$

This leads to a contradiction with (2); hence $T_1 \in \partial_w H_1(u; \theta)$.

Similarly, $T_1 \in -\partial_w H_2(u; \theta)$.

By the definition of the weak subdifferential, it is easy to see that $T_1 \in \partial_w H_1(u; \theta)$ if and only if $-T_1(u) \in w - \min \bigcup_{v\in E} \{H_1(v) - T_1(v)\}$, if and only if $-T_1(u) \in w - \min \bigcup_{v\in E} \{F_1(z) - z_1 - T(v - u) - T_1(v)\}$, if and only if $z_1 - (T + T_1)(u) \in w - \min \bigcup_{v\in E} \{F_1(v) - (T + T_1)(v)\}$.

This shows that $T + T_1 \in \partial_w F_1(u; z_1)$. Hence $T_1 \in \partial_w F_1(u; z_1) - T$. Similarly, $-T_1 \in \partial_w H_2(u; \theta)$ implies $-T_1 \in \partial_w F_2(u; z_2)$. Thus, $\theta = T_1 + (-T_1) \in \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2) - T$ and $T \in \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2)$. Therefore $\partial_w(F_1 + F_2)(u; z_1 + z_2) \subseteq \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2)$ and the proof is completed.

COROLLARY 3.2. *In Theorem* 3.1 *if* $Y = \mathbb{R}$, *then we have*

$$\partial_w(F_1 + F_2)(u; z_1 + z_2) = \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2).$$

Proof. By Theorem 3.1 we have $\partial_w(F_1 + F_2)(u; z_1 + z_2) \subseteq \partial_w F_1(u; z_1) + \partial_w F_2(u; z_2)$. The inclusion $\partial_w F_1(u; z_1) + \partial_w F_2(u; z_2) \subseteq \partial_w(F_1 + F_2)(u; z_1 + z_2)$ follows immediately from the definition of weak subdifferential and the proof of the corollary is completed.

Remark 1. If F_1 and F_2 are real and single valued functions, then Corollary 3.2 reduces to the Moreau–Rockafellar theorem [11].

Remark 2. If $Y \neq \mathbb{R}$, the inclusion $\partial_w F_1(u; z_1) + \partial_w F_2(u; z_2) \subseteq \partial_w (F_1 + F_2)(u; z_1 + z_2)$ may not be true.

EXAMPLE [15]. Let $X = \mathbb{R}^1$, $Y = \mathbb{R}^2$, and $C = \mathbb{R}^2_+$, $g_1(x) = (-x, 2x)$, $g_2(x) = (2x, -x)$; then g_1, g_2: $\mathbb{R}^1 \to \mathbb{R}^2$ are continuous, $\mathbb{R}^2_+$-convex functions and $g_1(0) = g_2(0) = 0$.

It is easy to see

$$\begin{aligned}\partial_w g_1(0; (0, 0)) &= \{(a, b) \in \mathbb{R}^2 \mid (0, 0) \in w - \min\{(-x, 2x) \\ &\qquad - (ax, bx), x \in \mathbb{R}\}\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid a \leq -1, b \geq 2 \text{ or } a \geq -1, b \leq 2\}.\end{aligned}$$

Similarly

$$\begin{aligned}\partial_w g_2(0; (0, 0)) &= \{(a, b) \in \mathbb{R}^2 \mid (0, 0) \in w - \min\{(2x, -x) \\ &\qquad - (ax, bx), x \in \mathbb{R}\}\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid a \leq 2, b \geq -1 \text{ or } a \geq 2, b \leq -1\}.\end{aligned}$$

Next

$$(g_1 + g_2)(x) = (x, x), \qquad (g_1 + g_2)(0) = (0, 0).$$

$$\begin{aligned}\partial_w (g_1 + g_2)(0; (0, 0)) &= \{(a, b) \in \mathbb{R}^2 \mid (0, 0) \in w \\ &\qquad - \min\{(x, x) - (ax, bx) \mid x \in \mathbb{R}\}\} \\ &= \{(a, b) \in \mathbb{R}^2 \mid a \geq 1, b \leq 1 \text{ or } a \leq 1, b \geq 1\}.\end{aligned}$$

It is obvious that $\partial_w g_1(0; (0, 0)) + \partial_w g_2(0; (0, 0)) = \mathbb{R}^2 \supsetneqq \partial_w (g_1 + g_2)(0; (0, 0))$.

THEOREM 3.3. *(Generalized Farkas–Minkowski theorem for set-valued functions). Let E be a convex subset of X. If the set-valued function F: $E \to 2^Y$ is C-convex, G: $E \to 2^Z$ is D-convex, and the system*

$$\begin{cases} F(x) <_C \theta \\ G(x) <_D \theta \end{cases}$$

has no solution in E, then there exists $(y^*, z^*) \neq (\theta, \theta)$ in $C^* \times D^*$ such that for $x \in E$

$$\langle y^*, F(x) \rangle + \langle z^*, G(x) \rangle \geq 0; \tag{3}$$

i.e., $\langle y^*, y \rangle + \langle z^*, z \rangle \geq 0$ for all $y \in F(x)$, $z \in G(x)$.

Proof. Let $A = \{(y, z) \in Y \times Z \mid$ there exists $x \in E$ such that $u <_C y$ and $v <_D z$ for some $u \in F(x)$ and $v \in G(x)\}$.

It is obvious that A does not contain origin (θ, θ) of $Y \times Z$.

We wish to show that A is convex in $Y \times Z$.

Let (y, z) and $(\bar{y}, \bar{z})$ be in A; then there exists $x_1, x_2 \in E$ such that

$$u_1 <_C y, \qquad v_1 <_D z, \qquad \text{and} \qquad u_2 <_C \bar{y},\ v_2 <_D \bar{z}$$

for some $u_1 \in F(x_1)$, $v_1 \in G(x_1)$, $u_2 \in F(x_2)$, and $v_2 \in G(x_2)$.

By the convexity of F and G, we have

$$\lambda F(x_1) + (1 - \lambda)F(x_2) \subset F[\lambda x_1 + (1 - \lambda)x_2] + C,$$

$$\lambda G(x_1) + (1 - \lambda)G(x_2) \subset G[\lambda x_1 + (1 - \lambda)x_2] + D.$$

Therefore there exist $c \in C$ and $d \in D$ and $a \in F[\lambda x_1 + (1 - \lambda)x_2]$, $b \in G[\lambda x_1 + (1 - \lambda)x_2]$ such that

$$\lambda u_1 + (1 - \lambda)u_2 = a + c, \qquad \lambda v_1 + (1 - \lambda)v_2 = b + d.$$

Hence

$$a = \lambda u_1 + (1 - \lambda)u_2 - c \leqq_C \lambda u_1 + (1 - \lambda)u_2 <_C \lambda y + (1 - \lambda)\bar{y},$$

$$b = \lambda v_1 + (1 - \lambda)v_2 - d \leqq_D \lambda v_1 + (1 - \lambda)v_2 <_D \lambda z + (1 - \lambda)\bar{z}.$$

This shows that $\lambda(y, z) + (1 - \lambda)(\bar{y}, \bar{z}) \in A$ and A is convex in $Y \times Z$.

Since int $C \neq \varnothing$ and int $D \neq \varnothing$, it follows that A has a non-empty interior. By the separation theorem there exists a nonzero element $(y^*, z^*) \in Y^* \times Z^*$ such that

$$\langle y^*, y \rangle + \langle z^*, z \rangle \geq 0 \qquad \text{for all } (y, z) \in A. \tag{4}$$

If $x \in E$, and for any $y \in F(x)$, $z \in G(x)$ and any $c \in \text{int } C$, $d \in \text{int } D$ we have $(y + c, z + d) \in A$.

By (4), we get

$$\langle y^*, y \rangle + \langle z^*, z \rangle + \langle y^*, c \rangle + \langle z^*, d \rangle \geq 0 \qquad \text{for } c \in \text{int } C,\ d \in \text{int } D. \tag{5}$$

It is obvious that $(y^*, z^*) \in C^* \times D^*$.

Letting $c \to \theta$ and $d \to \theta$ in (5), we obtain

$$\langle y^*, y \rangle + \langle z^*, z \rangle \geq 0 \qquad \text{for all } y \in F(x),\ z \in G(x), \text{ and } x \in E.$$

Hence $\langle y^*, F(x) \rangle + \langle z^*, G(x) \rangle \geq 0$ for all $x \in E$.

COROLLARY 3.4. *In Theorem 3.3, if we assume further that there exists $\hat{x} \in E$ such that $G(\hat{x}) \cap (-D) \neq \emptyset$, then there exists a $W \in B^+(Z, Y)$ such that*

$$F(x) + W(G(x)) <_C \theta$$

does not hold for any $x \in E$.

Proof. It follows from Theorem 3.3 that there is a nonzero $(y^*, z^*) \in C^* \times D^*$ such that (3) holds. We wish to show that $y^* \neq 0$. Suppose $y^* = 0$; then $z^* \neq 0$ in D^*. Hence $\langle z^*, z\rangle > 0$ for all $z \in \text{int } D$.

Since by assumption $G(\hat{x}) \cap (-\text{int } D) \neq \emptyset$ for some $\hat{x} \in E$, it follows that there exists $z \in G(\hat{x}) \cap (-\text{int } D)$ and so $0 > \langle z^*, z\rangle = \langle y^*, F(\hat{x})\rangle + \langle z^*, z\rangle \geq 0$ is a contradiction. Hence $0 \neq y^* \in C^*$ and $\langle y^*, y\rangle > 0$ for all $y \in \text{int } C \neq \emptyset$.

Let $y_0 \in \text{int } C$ be such that $\langle y^*, y_0\rangle = 1$.

Define $W\colon Z \to Y$ by $W(z) = \langle z^*, z\rangle\, y_0$.

Then $W \in B^+(Z, Y)$ and

$$\langle y^*, F(x) + W(G(x))\rangle \geq 0 \qquad \text{for all } x \in E. \tag{6}$$

Since $\langle y^*, y\rangle > 0$ for all $y \in \text{int } C$, it follows from (6) that there does not exist $x \in E$ such that $F(x) + W(G(x)) <_C \theta$ and the proof is completed.

As a consequence of Theorem 3.3, we obtain a necessary conditions for the existence of weak minimal point of the problem (P). The following theorem is due to Corley [5]. However, we give a different proof.

THEOREM 3.5[5]. *Let E be a convex subset of X and $F\colon E \to 2^Y$, $G\colon E \to 2^Z$ be respectively C-convex and D-convex set-valued functions. If x_0 is a weak minimal point of the problem* (P) *and $u \in F(x_0)$, $u \in w - \min F[E \cap G^-(-D)]$, then there exists a nonzero (y_0^*, z_0^*) in $C^* \times D^*$ and $z_0 \in G(x_0) \cap (-D)$ such that $\langle z_0^*, z_0\rangle = 0$ and*

$$\langle y_0^*, F(x)\rangle + \langle z_0^*, G(x)\rangle \geq \langle y_0^*, u\rangle \qquad \text{for all } x \in E.$$

Proof. By assumption, $x_0 \in E \cap G^-(-D)$, $u \in F(x_0)$, and $u \in w - \min F[E \cap G^-(-D)]$.

We wish to show that the system

$$\begin{cases} F(x) - u <_C \theta \\ \quad G(x) <_D \theta \end{cases} \tag{7}$$

has no solution in E.

For if $\bar{x} \in E$ were a solution of (7), then $G(\bar{x}) <_D \theta$ and $F(\bar{x}) - u <_C \theta$.

This shows that there exists $\bar{x} \in E \cap G^{-}(-\text{int } D)$ such that $F(\bar{x}) <_C u$. This contradicts the assumption that x_0 is a weak minimal point of (P). Hence system (7) has no solution.

It follows from Theorem 3.3 that there exists a nonzero element $(y_0^*, z_0^*) \in C^* \times D^*$ such that

$$\langle y_0^*, F(x) - u\rangle + \langle z_0^*, G(x)\rangle \geq 0 \qquad \text{for all } x \in E. \tag{8}$$

Thus

$$\langle y_0^*, a\rangle + \langle z_0^*, G(x)\rangle \geq \langle y_0^*, u\rangle \qquad \text{for all } a \in F(x),\ x \in E. \tag{9}$$

Letting $x = x_0$ and $a = u$ in (9), we obtain

$$\langle z_0^*, G(x_0)\rangle \geq 0. \tag{10}$$

Since $x_0 \in E \cap G^{-}(-D)$, it follows that $G(x_0) \cap (-D) \neq \varnothing$.
Let

$$z_0 \in G(x_0) \cap (-D); \qquad \text{then } \langle z_0^*, z_0\rangle \leq 0. \tag{11}$$

It follows from (10), (11), $\langle z_0^*, z_0\rangle = 0.$ (12)
Therefore

$$\langle y_0^*, F(x)\rangle + \langle z_0^*, G(x)\rangle \geq \langle y_0^*, u\rangle \qquad \text{for all } x \in E.$$

Applying Theorem 3.5 and following the similar arguments of Theorem 3.4, we obtain Corollary 3.6.

COROLLARY 3.6[5]. *In Theorem 3.5, if we assume further that there exists $\hat{x} \in E$ such that $G(\hat{x}) \cap (-\text{int } D) \neq \varnothing$, then there exist $w_0 \in B^+(Z; Y)$ and $z_0 \in G(x_0) \cap (-D)$ such that $w_0(z_0) = \theta$ and x_0 is a weak minimal point for the problem*

$$w - \min \bigcup_{x \in E} F(x) + w_0(G(x)).$$

If $Y = \mathbb{R}$, $x^* \in B(X, \mathbb{R}) = X^*$, $F\colon E \to 2^{\mathbb{R}}$, $x_0 \in X$, and $y_0 \in F(x_0)$, we see from the definition of weak subdifferential, that

$$x^* \in \partial_w F(x_0; y_0) \qquad \text{if } F(x) - \langle x^*, x\rangle \geq y_0 - \langle x^*, x_0\rangle, \qquad \text{for all } x \in E.$$

If $F\colon X \to 2^{\mathbb{R}}$, we denote $\partial F(x_0; y_0) = \partial_w F(x_0; y_0)$.

The D-convex set-valued function $G\colon X \to 2^Z$ is said to be weakly regular subdifferentiable at x_0 if

$$\partial_w(w_0 \circ G)(x_0; w_0(z)) = w_0 \circ \partial_w G(x_0; z) \quad \text{for every } z \in G(x_0) \quad \text{and} \quad w_0 \in B^+(Z, Y).$$

Applying Theorems 3.1 and 3.5, we prove a Fritz John type theorem for the existence of weak minimal point to the problem (P) as follows:

THEOREM 3.7. *Let* $F: E \to 2^Y$ *be* C*-convex and* $G: E \to 2^Z$ *be* D*-convex set-valued functions, where* $\operatorname{dom} F = \operatorname{dom} G = E$ *is a convex subset of* X. *Assume that one of* F *and* G *is connected at* $x_0 \in \operatorname{int} E$. *If* x_0 *is a weak minimal point to* (P), *and* $u \in w - \min F[E \cap G^-(-D)]$, *then there exist* $(y^*, z^*) \in C^* \times D^*$ *and* $z_0 \in G(x_0) \cap (-D)$, *such that* $\langle z^*, z_0\rangle = 0$ *and*

$$0 \in \partial y^* \circ F(x_0; \langle y^*, u\rangle) + \partial z^* \circ G(x_0; \langle z^*, z_0\rangle).$$

Proof. Assume x_0 is a weak minimal point to (P) and $u \in F(x_0)$, $u \in w - \min F[E \cap G^-(-D)]$.

It follows from Theorem 3.5 that there exist $(y^*, z^*) \in C^* \times D^*$ and $z_0 \in G(x_0) \cap (-D)$ such that $\langle z^*, z_0\rangle = 0$ and

$$\langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle \geq \langle y^*, u\rangle + \langle z^*, z_0\rangle \qquad \text{for all } x \in E. \quad (13)$$

Because $\langle y^*, u\rangle + \langle z^*, z_0\rangle \in \langle y^*, F(x_0)\rangle + \langle z^*, G(x_0)\rangle$ it follows from (13) that x_0 is a minimal solution of the following problem:

$$\min \bigcup_{x\in E} \langle y^*, F(x)\rangle + \langle z^*, G(x)\rangle.$$

Hence $0 \in \partial_w(y^* \circ F(x) + z^* \circ G(x))(x_0; \langle y^*, u\rangle + \langle z^*, z_0\rangle)$.

It follows from Theorem 3.1 that

$$0 \in \partial y^* \circ F(x_0; \langle y^*, u\rangle) + \partial z^* \circ G(x_0; \langle z^*, z_0\rangle).$$

Thus the proof of the theorem is completed.

In Theorem 3.7, if we add a further condition, we obtain a Kuhn–Tucker type necessary condition for the existence of weak minimal of the problem (P).

THEOREM 3.8. *Let* $E = \operatorname{dom} F = \operatorname{dom} G$ *be a convex subset of* X, *and* $F: E \to 2^Y$ *and* $G: E \to 2^Z$ *be respectively* C*-convex and* D*-convex set-valued functions. Assume that one of* F *and* G *is connected at* $u_0 \in \operatorname{int} E$ *and there exists* $\hat{x} \in E$ *such that* $G(\hat{x}) \cap (-\operatorname{int} D) \neq \emptyset$. *If* x_0 *is a weak minimal point to the problem* (P), G *is weakly regular subdifferentiable at* x_0, $u \in F(x_0)$, *and* $u \in w - \min F[E \cap G^-(-D)]$, *then there exist* $w_0 \in B^+(Z, Y)$ *and* $z_0 \in G(x_0) \cap (-D)$ *such that*

$$w_0(z_0) = \theta \quad \text{and} \quad 0 \in \partial_w F(x_0; u) + w_0 \circ \partial_w G(x_0; z_0).$$

Proof. Applying Theorem 3.1 and Corollary 3.6 and following the arguments of Theorem 3.7, this theorem follows immediately.

COROLLARY 3.9. *Let E be a convex subset of X and F, G: $E \to 2^{\mathbb{R}}$ be $\mathbb{R}_+$-convex. Assume that one of F and G is connected at some $u_0 \in \text{int } E$ and there exists $\hat{x} \in E \cap G^-(\text{int } \mathbb{R}_-)$. If x_0 is a minimal solution to (P), then there exists $\lambda \geq 0$, $z_0 \in G(x_0) \cap (\mathbb{R}_-)$, such that $\lambda_0 z_0 = 0$ and $0 \in \partial F(x_0; u) + \lambda \partial G(x_0; z_0)$ where $u \in F(x_0)$ and $u = w - \min \bigcup_{x \in E \cap G^-(-\mathbb{R}_+)} F(x)$.*

Proof. Since G is a real set-valued function, it follows that $\partial \lambda G(x_0; \lambda z_0) = \lambda \partial G(x_0, z)$ for all $\lambda \geq 0$. Hence G is a weakly regular subdifferentiable and the corollary follows immediately from Theorem 3.8.

LEMMA 3.10. *Let F: $E \to 2^{\mathbb{R}}$ be strictly $\mathbb{R}_+$-convex on a convex subset E of X. Then, for $x_0 \in E$, $y_0 \in F(x_0)$, and $x^* \in \partial F(x_0; y_0)$, we have*

$$F(x) > y_0 + \langle x^*, x - x_0 \rangle \quad \text{for all } x \neq x_0 \text{ in } E.$$

Proof. Since F: $E \to 2^{\mathbb{R}}$ is strictly $\mathbb{R}_+$-convex on E, we have

$$\lambda F(x) + (1 - \lambda)F(x_0) \subseteq F[\lambda x + (1 - \lambda)x_0] + \text{int } \mathbb{R}_+ \quad \text{for any } \lambda \in (0,1) \quad \text{and} \quad x \neq x_0.$$

Let $y \in F(x)$; then $\lambda y + (1 - \lambda)y_0 \in F[\lambda x + (1 - \lambda)x_0] + \text{int } \mathbb{R}_+$.

Hence there exists $u \in F[\lambda x + (1 - \lambda)x_0]$ such that $\lambda y + (1 - \lambda)y_0 = u + c$ for some $c > 0$.

Therefore $u = \lambda y + (1 - \lambda)y_0 - c$.

For $x^* \in \partial F(x_0; y_0)$, we have

$$F[\lambda x + (1 - \lambda)x_0] - y_0 \geq \langle x^*, \lambda x + (1 - \lambda)x_0 - x_0 \rangle = \lambda \langle x^*, x - x_0 \rangle.$$

This implies $u - y_0 \geq \lambda \langle x^*, x - x_0 \rangle$.

But $\lambda(y - y_0) - c = \lambda y + (1 - \lambda)y_0 - c - y_0 = u - y_0 \geq \lambda \langle x^*, x - x_0 \rangle$.

This implies $y - y_0 \geq \langle x^*, x - x_0 \rangle + c/\lambda > \langle x^*, x - x_0 \rangle$ for all $y \in F(x)$ and $\lambda \in (0, 1)$.

Therefore $F(x) - y_0 > \langle x^*, x - x_0 \rangle$, for all $x \neq x_0$ in E.

Following the arguments of [8], we prove Theorems 3.11 and 3.12.

THEOREM 3.11. *Let E be a convex subset of X, and let $F = (F_1, \ldots, F_n)$: $E \to 2^{\mathbb{R}^n}$ and G: $E \to 2^{\mathbb{R}^m}$ be respectively* $\mathbb{R}^n_+$-convex and *$\mathbb{R}^m_+$-convex set-valued functions. If $x_0 \in E \cap G^-(\mathbb{R}^m_-)$ is a Geoffrion efficient solution, then there exists $u = (u_1, \ldots, u_n)$ with strictly positive components such that x_0 is an optimal solution to the problem (P_u), where*

$$\min \sum_{i=1}^{n} u_i F_i(x), \text{ subject to } x \in E,\ G(x) \cap \mathbb{R}^m_- \neq \varnothing. \tag{P$_u$}$$

Proof. If x_0 is a Geoffrion efficient solution, then there exist $y = (y_1, \ldots, y_n) \in F(x_0)$ and $M > 0$ such that for each $i = 1, \ldots, n$, the system

$$\begin{cases} y_i > F_i(x) \\ y_i > F_i(x) + M(F_j(x) - y_j), \qquad i \neq j \end{cases} \tag{14}$$

has no solution in $E \cap G^-(\mathbb{R}^m_-)$.

By Theorem 3.3, for the ith system, there exists $\lambda^i_j \geq 0$ with $\sum_{j=1}^{n} \lambda^i_j = 1$ such that

$$\lambda^i_i(F_i(x) - y_i) + \sum_{j \neq i} \lambda^i_j(F_i(x) + MF_j(x) - y_i - My_j) \geq 0. \tag{15}$$

Since

$$\left(\lambda^i_i + \sum_{j \neq i} \lambda^i_j\right) F_i(x) \subseteq \lambda^i_i F_i(x) + \sum_{j \neq i} \lambda^i_j F_i(x) \tag{16}$$

it follows from (15) and (16) that

$$F_i(x) + M \sum_{j \neq i} \lambda^i_j F_j(x) - y_i - M \sum_{j \neq i} \lambda^i_j y_j \geq 0. \tag{17}$$

Summing up the above n inequalities, we have

$$\sum_{i=1}^{n} F_i(x) + M \sum_{i=1}^{n} \sum_{i \neq j} \lambda^i_j F_j(x) - \sum_{i=1}^{n} y_i - M \sum_{i=1}^{n} \sum_{i \neq j} \lambda^i_j y_j \geq 0.$$

Since

$$\left(1 + M \sum_{i \neq j} \lambda^j_i\right) F_i(x) \subseteq F_i(x) + M \sum_{i \neq j} \lambda^j_i F_i(x)$$

we have

$$\sum_{j=1}^{n} \left(1 + M \sum_{i \neq j} \lambda^i_j\right) F_j(x) \geq \sum_{j=1}^{n} \left(1 + M \sum_{i \neq j} \lambda^i_j\right) y_j \tag{18}$$

for all $x \in E \cap G^-(\mathbb{R}^m_-)$. Since

$$\sum_{j=1}^{n}\left(1 + M\sum_{i\neq j}\lambda_j^i\right) y_j \in \sum_{j=1}^{n}\left(1 + M\sum_{i\neq j}\lambda_j^i\right) F_j(x_0), \tag{19}$$

(18) and (19) show that x_0 is a solution of (P_u) with $u = (1 + M\sum_{i\neq 1}\lambda_1^i, \ldots, 1 + M\sum_{i\neq n}\lambda_n^i)$.

THEOREM 3.12. *Let $u_i > 0$ ($i = 1, \ldots, n$) be fixed; if x_0 is an optimal solution to the problem (P_u), then x_0 is a Geoffrion efficient solution to the problem* (P).

Proof. If x_0 is an optimal solution to the problem (P_u), then there exists $y = (y_1, \ldots, y_n) \in F(x_0)$ such that $\sum_{i=1}^n u_i y_i = \min \bigcup_{x\in E\cap G^-(\mathbb{R}^m_-)} \sum_{i=1}^n u_i F_i(x)$. Hence

$$\sum_{i=1}^{n} u_i y_i \leq \sum_{i=1}^{n} u_i F_i(x) \qquad \text{for all } x \in E \cap G^-(\mathbb{R}^m_-). \tag{20}$$

It is obvious that

$$y \in \underline{\text{eff}} \bigcup_{x\in E\cap G^-(\mathbb{R}^m_-)} F(x).$$

Therefore x_0 is a lower efficient solution to the problem (P).

We wish to show that x_0 is a Geoffrion efficient solution to (P) with $M = (n-1)\max_{i,j} u_j/u_i$. Suppose, to the contrary, that for some $x \in E \cap G^-(\mathbb{R}^m_-)$, and some i and some $w_i \in F_i(x)$ with $w_i < y_i$, we have $y_i - w_i > M(w_j - y_j)$ for all j with $w_j \in F_j(x)$ and $w_j > y_j$. It follows that

$$y_i - w_i > \frac{n-1}{u_i} u_j(w_j - y_j) \qquad \text{for all } j \neq i.$$

Multiplying through by $u_i/n - 1$ and summing over $j \neq i$ yields

$$u_i y_i - u_i w_i > \sum_{j\neq i} u_j(w_j - y_j).$$

It follows that $\sum u_i y_i > \sum u_j w_j$ for $x \in E \cap G^-(\mathbb{R}^m_-)$ and some $(w_1, \ldots, w_n) \in F(x) = (F_1(x), \ldots, F_n(x))$, which contradicts (20).

We complete the proof of Theorem 3.12.

A set $A \subseteq \mathbb{R}^n$ is called $\mathbb{R}^n_+$-convex if $A + \mathbb{R}^n_+$ is a convex set in $\mathbb{R}^n$.

LEMMA 3.13. *Let E be a convex subset of X, and $F: E \to 2^{\mathbb{R}^n}$ and $G: E \to 2^{\mathbb{R}^m}$ be respectively $\mathbb{R}^n_+$-convex and $\mathbb{R}^m_+$-convex set-valued functions; then $F[E \cap G^-(\mathbb{R}^m_-)]$ is a $\mathbb{R}^n_+$-convex set in $\mathbb{R}^n$.*

Proof. If $x_1, x_2 \in E \cap G^-(\mathbb{R}^m_-)$, $\lambda \in [0,1]$, then $x_1, x_2 \in E$, $G(x_1) \cap \mathbb{R}^m_- \neq \emptyset$, $G(x_2) \cap \mathbb{R}^m_- \neq \emptyset$. We choose $z_1 \in G(x_1) \cap \mathbb{R}^m_-$, $z_2 \in G(x_2) \cap \mathbb{R}^m_-$. By the $\mathbb{R}^m_+$-convexity of G, we have

$$\lambda G(x_1) + (1 - \lambda)G(x_2) \subset G[\lambda x_1 + (1 - \lambda)x_2] + \mathbb{R}^m_+.$$

Hence there exists $z \in G[\lambda x_1 + (1 - \lambda)x_2]$, $b \in \mathbb{R}^m_+$ such that

$$z + b = \lambda z_1 + (1 - \lambda)z_2.$$

Thus

$$z = \lambda z_1 + (1 - \lambda)z_2 - b \in \mathbb{R}^m_- \cap G[\lambda x_1 + (1 - \lambda)x_2] \neq \emptyset.$$

Therefore $\lambda x_1 + (1 - \lambda)x_2 \in E \cap G^-(\mathbb{R}^m_-)$ and $E \cap G^-(\mathbb{R}^m_-)$ is a convex set in X. Next let $y_1, y_2 \in F[E \cap G^-(\mathbb{R}^m_-)]$, $\lambda \in [0,1]$; then there exists $x_1, x_2 \in E \cap G^-(\mathbb{R}^m_-)$ such that $y_1 \in F(x_1)$, $y_2 \in F(x_2)$. By the $\mathbb{R}^n_+$-convexity of F, we have

$$\begin{aligned}\lambda y_1 + (1 - \lambda)y_2 \in \lambda F(x_1) + (1 - \lambda)F(x_2) &\subseteq F[\lambda x_1 + (1 - \lambda)x_2] + \mathbb{R}^n_+ \\ &\subseteq F[E \cap G^-(\mathbb{R}^m_-)] + \mathbb{R}^n_+.\end{aligned}$$

This shows that $F[E \cap G^-(\mathbb{R}^m_-)]$ is $\mathbb{R}^n_+$-convex.

LEMMA 3.14[13]. *Let A be a $\mathbb{R}^n_+$ convex set; then $y_0 \in A$ satisfies*

$$\overline{A + \mathbb{R}^n_+ - y_0} \cap (\mathbb{R}^n_-) = \{0\}$$

iff there exists a vector $\hat{v} \in \operatorname{int} \mathbb{R}^n_+$ such that

$$\langle \hat{v}, y_0 \rangle \leq \langle \hat{v}, y \rangle \qquad \text{for all } y \in A.$$

THEOREM 3.15. *Let E be a convex subset of X, and F: $E \to 2^{\mathbb{R}^n}$ and G: $E \to 2^{\mathbb{R}^m}$ be respectively $\mathbb{R}^n_+$-convex and $\mathbb{R}^m_+$-convex set-valued functions. Then x_0 is a Geoffrion efficient solution of the problem* (P) *iff x_0 is a proper efficient solution of the problem* (P).

Proof. Theorem 3.15 follows immediately from Definition 6, Theorems 3.11 and 3.12, and Lemmas 3.13 and 3.14.

Theorem 3.16 establishes a sufficient condition for the existence of Geoffrion efficient solution of the problem (P).

THEOREM 3.16. *Let E be a convex subset of X, and $F = (F_1, \ldots, F_n)$: $E \to 2^{\mathbb{R}^n}$ and G: $E \to 2^{\mathbb{R}^m}$ be respectively $\mathbb{R}^n_+$-convex and strictly $\mathbb{R}^m_-$-convex set-valued functions. Suppose that $x_0 \in E$ and that there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in \operatorname{int} \mathbb{R}^n_+$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^m_+$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_m) \in G(x_0) \cap (\mathbb{R}^m_-)$,*

$\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in F(x_0)$ *such that* $\langle \beta, \bar{z} \rangle = 0$ *and* $0 \in \sum_{i=1}^{n} \lambda_i \partial F_i(x_0; \bar{y}_i) + \sum_{j=1}^{m} \beta_j \partial G_j(x_0; \bar{z}_j)$. *Then* x_0 *is a Geoffrion efficient solution of the problem* (P).

Proof. By assumption, for each $i = 1, \ldots, n$, and $j = 1, \ldots, m$, there exist $x_i^* \in \partial F_i(x_0; y_i^*)$ and $z_j^* \in \partial G_j(x_0; z_j^*)$ such that $0 = \sum_{i=1}^{n} \lambda_i x_i^* + \sum_{j=1}^{m} \beta_j z_j^*$. Let x be any feasible solution of the problem (P); then $G(x) \cap (\mathbb{R}_-^m) \neq \emptyset$, and we choose $z = (z_1, \ldots, z_m) \in G(x) \cap (\mathbb{R}_-^m)$. By assumption, for each $j = 1, \ldots, m$, G_j is strictly $\mathbb{R}_+$-convex, and it follows from Lemma 3.10 that

$$G_j(x) - \bar{z}_j > \langle z_j^*, x - x_0 \rangle, \qquad j = 1, \ldots, m.$$

Hence $z_j - \bar{z}_j > \langle z_j^*, x - x_0 \rangle, j = 1, \ldots, m$. Since $\beta \in \mathbb{R}_+^m$, $\beta \neq 0$, we have

$$\begin{aligned} 0 \geq \langle \beta, z \rangle &= \langle \beta, z \rangle - \langle \beta, \bar{z} \rangle \\ &= \langle \beta, z - \bar{z} \rangle = \sum_{j=1}^{m} \beta_j (z_j - \bar{z}_j) \\ &> \sum_{j=1}^{m} \beta_j \langle z_j^*, x - x_0 \rangle = \sum_{j=1}^{m} \langle \beta_j z_j^*, x - x_0 \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} \langle \lambda, F(x) - \bar{y} \rangle &= \sum_{i=1}^{n} \lambda_i (F_i(x) - \bar{y}_i) \geq \sum_{i=1}^{n} \lambda_i \langle x_i^*, x - x_0 \rangle \\ &= -\sum_{j=1}^{n} \beta_j \langle z_j^*, x - x_0 \rangle > 0. \end{aligned}$$

From this

$$\langle \lambda, \bar{y} \rangle < \langle \lambda, F(x) \rangle$$

for all feasible solutions x of problem (P) and the theorem follows immediately from Theorem 3.12.

Applying Theorems 3.11 and 3.5, and Lemma 3.1, we obtain a necessary condition for the existence of the Geoffrion efficient solution of the problem (P).

THEOREM 3.17. *Let E be a convex subset of X, and* $F = (F_1, \ldots, F_n)$: $E \to 2^{\mathbb{R}^n}$ *and* $G = (G_1, \ldots, G_m)$: $E \to 2^{\mathbb{R}^m}$ *be respectively* $\mathbb{R}_+^n$*-convex and* $\mathbb{R}_+^m$*-convex set-valued functions. Assume that all* $F_1, \ldots, F_n, G_1, \ldots, G_m$ *except possibly one are connected at some* $u_0 \in \operatorname{int} E$ *and there exists* $\hat{x} \in E \cap$

$G^{-}(\text{int } \mathbb{R}^m_-)$. *If* x_0 *is a Geoffrion efficient solution of the problem* (P) *and* $y = (y_1, \ldots, y_n) \in F(x_0)$ *is a Geoffrion efficient value, then there exist* $\lambda = (\lambda_1, \ldots, \lambda_n) \in \text{int } \mathbb{R}^m_-$, $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+$, $\bar{z} = (z_1, \ldots, z_m)$, $\bar{z} \in G(x_0) \cap (\mathbb{R}^m_-)$ *such that* $\langle \beta, \bar{z} \rangle = 0$ *and*

$$0 \in \sum_{i=1}^{n} \lambda_i \partial F_i(x_0; y_j) + \sum_{j=1}^{m} \beta_j \partial G_j(x_0; z_j).$$

Proof. Given that x_0 is a Geoffrion efficient solution of the problem (P), it follows from Theorem 3.11 that there exist $\lambda = (\lambda_1, \ldots, \lambda_n) \in \text{int } \mathbb{R}^n_+$ such that

$$\langle \lambda, y \rangle \leq \langle \lambda, F(x) \rangle \qquad \text{for all } x \in E \cap G^{-}(\mathbb{R}^m_-).$$

That is, x_0 is an optimal solution of the problem

$$\text{Min} \sum_{i=1}^{n} \lambda_i F_i(x), \qquad \text{subject to} \qquad x \in E,\ G(x) \cap (\mathbb{R}^m_-) \neq \varnothing. \tag{Q}$$

Then, by Theorem 3.5, there exists $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^m_+$ and $\bar{z}_0 \in G(x_0) \cap (\mathbb{R}^m_-)$ such that $\langle \beta, z_0 \rangle = 0$ and x_0 is an optimal solution of the problem

$$\text{Min} \bigcup_{x \in E} \sum_{i=1}^{n} \lambda_i F_i(x) + \sum_{j=1}^{m} \beta_j G_j(x). \tag{Q'}$$

Hence $0 \in \partial(\sum_{i=1}^{n} \lambda_i F_i + \sum_{j=1}^{m} \beta_j G_j)(x_0; \sum_{i=1}^{n} \lambda_i y_i + \sum_{j=1}^{m} \beta_j z_j)$.
Since $\partial \lambda_i F_i(x_0; \lambda_i y_i) = \lambda_i \partial F_i(x_0; y_i)$, it follows from Theorem 3.1 that

$$0 \in \sum_{i=1}^{n} \lambda_i \partial F_i(x_0; y_i) + \sum_{j=1}^{m} \beta_j \partial G_j(x_0; z_j),$$

and the proof of the theorem is completed.

4. Duality Theorem of Set-Valued Functions

In this section, we present and prove the Wolfe and Mond–Weir duality theorems of vector, set-valued functions. In this section, we assume that $Y = \mathbb{R}^n$ and $Z = \mathbb{R}^m_+$ in problem (P).

The Mond–Weir type duality problem of (P) is as follows:

$$\begin{aligned} &\text{Maximize } y, \text{ subject to } x \in E,\ \lambda = (\lambda_1, \ldots, \lambda_n) \in \operatorname{int} \mathbb{R}^n_+, \\ &\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+,\ y = (y_1, \ldots, y_n) \in F(x), \qquad \text{(D1)} \\ &z = (z_1, \ldots, z_m) \in G(x),\ \langle \beta, z\rangle \geq 0, \\ &0 \in \sum_{i=1}^{n} \lambda_i \partial F_i(x; y_i) + \sum_{j=1}^{m} \beta_j \partial G_j(x; z_j). \end{aligned}$$

The Wolfe type duality problem of (P) is as follows:

$$\text{Maximize } y + \langle \beta, z\rangle e, \text{ subject to } x \in E,\ \lambda = (\lambda_1, \ldots, \lambda_n) \in \operatorname{int} \mathbb{R}^n_+,$$

$$\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+,\ y = (y_1, \ldots, y_n) \in F(x),\ z = (z_1, \ldots, z_m) \in G(x),$$

$$0 \in \sum_{i=1}^{n} \lambda_i \partial F_i(x; y_i) + \sum_{j=1}^{m} \beta_j \partial G_j(x; z_j),\ e = (1, \ldots, 1) \in \mathbb{R}^n.$$

We say $(x, \lambda, \beta, y, z)$ is a feasible solution to (D1). If $\lambda \in \operatorname{int} \mathbb{R}^n_+$, $\beta \in \mathbb{R}^m_+$, $x \in E$, $y \in F(x)$, $z \in G(x)$, $\langle \beta, z\rangle \geq 0$ and $0 \in \sum_{i=1}^n \lambda_i \partial F_i(x; y_i) + \sum_{j=1}^m \beta_j \partial G_j(x; z_j)$. The feasible solution to (D2) can be defined similarly.

THEOREM 4.1. *(Weak duality theorem) Let x_0 be a feasible solution to* (P) *and $(x, \lambda, \beta, y, z)$ be a feasible solution to (D1); then $\langle \lambda, F(x_0)\rangle \geq \langle \lambda, y\rangle$.*

Proof. By assumption, there exist $x_i^* \in \partial F_i(x; y_i)$, $z_j^* \in \partial G_j(x; z_j)$, $j = 1, \ldots, m$, $i = 1, \ldots, n$ such that

$$0 = \sum_{i=1}^{n} \lambda_i x_i^* + \sum_{j=1}^{m} \beta_j z_j^*.$$

Since x_0 is feasible to (P), it follows that $G(x_0) \cap (\mathbb{R}^m_-) \neq \emptyset$, and we choose $\bar{z} \in G(x_0) \cap \mathbb{R}^m_-$. Then

$$0 \geq \langle \beta, \bar{z}\rangle - \langle \beta, z\rangle = \sum_{j=1}^{m} \beta_j(\bar{z}_j - z_j) \geq \sum_{j=1}^{m} \beta_j \langle z_j^*, x_0 - x\rangle.$$

Since $\langle \sum_{i=1}^{n} \lambda_i x_i^* + \sum_{j=1}^{m} \beta_j z_j^*, x_0 - x \rangle = 0$, we have $\langle \sum_{j=1}^{n} \lambda_j x_j^*, x_0 - x \rangle \geq 0$.

$$\langle \lambda, F(x_0) \rangle - \langle \lambda, y \rangle = \sum_{j=1}^{n} \lambda_j (F_j(x_0) - y_j) \geq \sum_{j=1}^{n} \lambda_j \langle x_j^*, x_0 - x \rangle$$

$$= \left\langle \sum_{j=1}^{n} \lambda_j x_j^*, x_0 - x \right\rangle \geq 0.$$

Therefore $\langle \lambda, F(x_0) \rangle \geq \langle \lambda, y \rangle$.

THEOREM 4.2. *(Duality theorem) Suppose that E is a convex subset of X, and $F = (F_1, \ldots, F_n)$: $E \to 2^{\mathbb{R}^n}$ and $G = (G_1, \ldots, G_m)$: $E \to 2^{\mathbb{R}^m}$ are respectively $\mathbb{R}^n_+$-convex and $\mathbb{R}^m_+$-convex, $F_1, \ldots, F_n, G_1, \ldots, G_m$ except possibly one are connected at $u_0 \in \operatorname{int} E$ and there exists $\hat{x} \in E$ such that $G(\hat{x}) \cap (\operatorname{int} \mathbb{R}^m_-) \neq \emptyset$. If x_0 is a Geoffrion efficient solution of the problem* (P), *then there exists $\bar{\lambda} \in \operatorname{int} \mathbb{R}^n_+$, $\bar{\beta} \in \mathbb{R}^m_+$, such that $(x_0, \bar{\lambda}, \bar{\beta}, \bar{y}, \bar{z})$ solves* (D1) *for some $\bar{y} \in F(x_0)$, $\bar{z} \in G(x_0)$. Furthermore, problems* (P) *and* (D1) *have the same extreme values.*

Proof. Since x_0 is a Geoffrion efficient solution of (P) and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in F(x_0)$ is a Geoffrion efficient value of (P), it follows from Theorem 3.17 that there exist $\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \operatorname{int} \mathbb{R}^n_+$, $\bar{\beta} = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m_+$, $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_m) \in G(x_0) \cap \mathbb{R}^m_-$ such that $\langle \bar{\beta}, \bar{z} \rangle = 0$ and

$$0 \in \sum_{i=1}^{n} \bar{\lambda}_i \partial F_i(x_0; \bar{y}_i) + \sum_{j=1}^{m} \bar{\beta}_j \partial G_j(x_0; \bar{z}_j).$$

In other words, $(x_0, \bar{\lambda}, \bar{\beta}, \bar{y}, \bar{z})$ is a feasible solution for (D1). Hence, if $(x, \lambda, \beta, y, z)$ is any feasible solution of (D1), then by Theorem 4.1, we have

$$\langle \lambda, \bar{y} \rangle \geq \langle \lambda, y \rangle.$$

Since $\lambda \in \operatorname{int} \mathbb{R}^m_+$, it follows that there does not exist y with $(x, \lambda, \beta, y, z)$ feasible solution of (D1) such that $y \geq \bar{y}$. This shows that $\bar{y}$ is an optimal value of (D1).

THEOREM 4.3. *(Weak duality theorem) Let x_0 be a feasible solution for* (P) *and $(x, \lambda, \beta, y, z)$ be a feasible solution for* (D2); *then $\langle \lambda, F(x_0) \rangle \geq \langle \lambda, y \rangle + \langle \beta, z \rangle$.*

Proof. By assumption, there exists $x_i^* \in \partial F_i(x, y_i)$, $z_j^* \in \partial G_j(x; z_j)$, $j = 1, \ldots, m$, $i = 1, \ldots, n$, such that $0 = \sum_{i=1}^{n} \lambda_i x_i^* + \sum_{j=1}^{m} \beta_j z_j^*$.

$$
\begin{aligned}
&\langle \lambda, F(x_0)\rangle - \langle \lambda, y\rangle - \langle \beta, z\rangle \\
&= \sum_{i=1}^{n} \lambda_i[F_i(x_0) - y_i] - \sum_{j=1}^{m} \beta_j z_j \\
&\geq \sum_{i=1}^{n} \lambda_i\langle y_i^*, x_0 - x\rangle - \sum_{j=1}^{m} \beta_j z_j \\
&= -\sum_{j=1}^{m} \beta_j\langle z_j^*, x_0 - x\rangle - \sum_{j=1}^{m} \beta_j z_j \\
&\geq \sum_{j=1}^{m} \beta_j\langle z_j - G(x_0)\rangle - \sum_{j=1}^{m} \beta_j z_j \\
&= -\langle \beta, G(x_0)\rangle
\end{aligned}
$$

Since x_0 is a feasible solution to problem (P), it follows that $G(x_0) \cap (\mathbb{R}^m_-) \neq \emptyset$. Choosing $z_0 \in G(x_0) \cap (\mathbb{R}^m_-)$, we see that

$$\langle \lambda, F(x_0)\rangle - \langle \lambda, y\rangle - \langle \beta, z\rangle \geq -\langle \beta, z_0\rangle \geq 0.$$

This shows that

$$\langle \lambda, F(x_0)\rangle \geq \langle \lambda, y\rangle + \langle \beta, z\rangle$$

and the theorem is completed.

THEOREM 4.4. *In Theorem* 4.2, *if* x_0 *is a Goeffrion efficient solution and* $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_n) \in F(x_0)$ *is a Geoffrion efficient value of the problem* (P), *then there exist* $\bar{\lambda} \in \text{int } \mathbb{R}^n_+$, $\beta \in \mathbb{R}^m_+$, *such that* $(x_0, \bar{\lambda}, \bar{\beta}, \bar{y}, \bar{z})$ *solves* (D2) *for some* $\bar{y} \in F(x_0)$, $\bar{z} \in G(x_0)$. *Furthermore, problems* (P) *and* (D2) *have the same extreme values.*

Proof. As in Theorem 4.2, there exist $\bar{\lambda} \in \text{int } \mathbb{R}^n_+$, $\bar{\beta} \in \mathbb{R}^m_+$, $\bar{y} \in F(x_0)$, $\bar{z} \in G(x_0) \cap \mathbb{R}^m_-$, such that $(x_0, \bar{\lambda}, \bar{\beta}, \bar{y}, \bar{z})$ is a feasible solution of (D2). Let $(x, \lambda, \beta, y, z)$ be any feasible solution of (D2).

Without loss of generality, we may assume that $\sum_{i=1}^{n} \lambda_i = 1$.

Since $\langle \bar{\beta}, \bar{z}\rangle = 0$, it follows from Theorem 4.3 that

$$\langle \lambda, \bar{y} + \langle \bar{\beta}, \bar{z}\rangle\, e\rangle = \langle \lambda, \bar{y}\rangle \geq \langle \lambda, y\rangle + \langle \beta, z\rangle = \langle \lambda, y + \langle \beta, z\rangle\, e\rangle.$$

Since $\lambda \in \text{int } \mathbb{R}^n_+$, there does not exist a feasible solution $(x, \lambda, \beta, y, z)$ for (D2) such that $y + \langle \beta, z\rangle\, e \geq \bar{y} = \bar{y} + \langle \bar{\beta}, \bar{z}\rangle e$. This shows that $\bar{y}$ is the extreme value of (D2).

THEOREM 4.5. *(Converse duality theorem) Let* $F: E \to 2^{\mathbb{R}}$ *be strictly* $\mathbb{R}_+$*-convex and* $G: E \to 2^{\mathbb{R}}$ *be* $\mathbb{R}_+$*-convex. Suppose one of F and G is connected at* $u_0 \in \operatorname{int} E$ *and there is a* $\hat{x} \in E$ *such that* $G(\hat{x}) \cap (\operatorname{int} \mathbb{R}_-) \neq \emptyset$. *If* $(x_0, 1, \beta_0, y_0, z_0)$ solves the problem (D2), *then* x_0 *solves* (P). *Moreover, the extreme values of problems* (P) *and* (Q) *are the same.*

Proof. Suppose that $\bar{x}$ solves problem (P). By Corollary 3.9 and Theorem 4.4 there exists $\bar{\beta} \geq 0, \bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x})$, such that $(\bar{x}, 1, \bar{\beta}, \bar{y}, \bar{z})$ is a solution of problem (D2) and $\bar{\beta}\bar{z} = 0$. Hence $\bar{y} + \bar{\beta}\bar{z} \geq y + \beta z$ for all feasible solutions $(x, 1, \beta, y, z)$ of (D2).

We wish to show that $\bar{x} = x_0$.

Since $(x_0, 1, \beta_0, y_0, z_0)$ solves problem (D2), it follows that $x_0 \in E$, $\beta_0 \geq 0$ and there exist $y_0 \in F(x_0)$, $z_0 \in G(x_0)$, $x^* \in \partial F(x_0; y_0)$, $z^* \in \partial G(x_0; z_0)$, such that $0 = x^* + \beta_0 z^*$ and $y_0 + \beta_0 z_0 = \bar{y} + \bar{\beta}\bar{z}$. If $x_0 \neq \bar{x}$, then by Lemma 3.17 we have

$$F(\bar{x}) - y_0 > \langle x^*, \bar{x} - x_0 \rangle. \tag{21}$$

But it follows from (21) and $\bar{y} \in F(\bar{x})$ that

$$\begin{aligned} \bar{y} - y_0 + \beta_0(G(\bar{x}) - z_0) &> \langle x^*, \bar{x} - x_0 \rangle + \beta_0 \langle z^*, \bar{x} - x_0 \rangle \\ &= \langle x^* + \beta_0 z^*, \bar{x} - x_0 \rangle = 0. \end{aligned}$$

That is,

$$\bar{y} + \beta_0 G(\bar{x}) > y_0 + \beta_0 z_0 = \bar{y} + \bar{\beta}\bar{z}.$$

Since $\bar{\beta}\bar{z} = 0$, it follows that

$$\beta_0 G(\bar{x}) > 0. \tag{22}$$

Since $\bar{x}$ solves problem (P), there exists $\bar{u} \in G(\bar{x}) \cap (\mathbb{R}_-) \neq \emptyset$ such that

$$\beta_0 \bar{u} \leq 0. \tag{23}$$

We see that (23) contradicts (22). Therefore $x_0 = \bar{x}$, so x_0 is a minimal solution for problem (P). The final part of the theorem follows immediately from Theorem 4.4.

THEOREM 4.6. *(Converse duality theorem) Let E be a convex subset of* $\mathbb{R}^n$, *and* $F: E \to \mathbb{R}^n$ *and* $G: E \to \mathbb{R}^m$ *be respectively* $\mathbb{R}^n_+$*-convex and* $\mathbb{R}^m_+$*-convex set-valued functions. Let* $(x_0, \lambda, \beta, y_0, z_0)$ *be a feasible solution for* (D1). *Suppose that there exists a feasible solution* $\bar{x}$ *for* (P) *and* $\bar{y} \in F(\bar{x})$ *such that* $\langle \lambda, \bar{y} \rangle = \langle \lambda, y_0 + \langle \beta, z_0 \rangle e \rangle$. *Then* $\bar{x}$ *is a proper efficient solution of* (P).

Proof. It follows from Theorem 4.3 that

$$\langle \lambda, F(x) \rangle \geq \langle \lambda, y_0 + \langle \beta, z_0 \rangle e \rangle = \langle \lambda, \bar{y} \rangle$$

for any feasible solution x of (P). The theorem follows immediately from Theorems 3.12 and 3.15.

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