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PRE-VECTOR VARIATIONAL INEQUALITIES

LAI-JIU LIN

Existence theorems for pre-vector variational inequalities are established under different conditions on the operator T and the function η. As an application, we establish the existence of a weak minimum of an optimisation problem on η-invex functions.

1. INTRODUCTION

Throughout this paper, let X, Z be Banach spaces, (Y, D) be an ordered Banach spaces, ordered by a closed convex cone D. Let $L(X,Y)$ be the space of all bounded linear operators from X to Y, $E \subseteq X$ and $C \subseteq Z$ be nonempty sets, $\eta : E \times E \to E$ be a function, $V : E \to 2^C$ and $G : E \to 2^E$ be set-valued maps. We consider the following three problems:

PRE-VVIP. Find $\overline{x} \in E$ such that

$$\langle T(\overline{x}), \eta(y, \overline{x})\rangle \not< 0 \text{ for all } y \in E,$$

where T is a map from E to $L(X,Y)$.

PRE-QVVIP. Find $\overline{x} \in E$, $\overline{y} \in V(\overline{x})$ such that

$$\langle H(\overline{x}, \overline{y}), \eta(y, \overline{x})\rangle \not< 0 \text{ for all } y \in G(\overline{x}),$$

where H is a map from $E \times C$ to $L(X,Y)$.

The Pre-VVIP has some relation with vector optimisation problems of η-invex function.

(P) V-min f(x) subject to $x \in E$,

where $f : E \to Y$ is a η-invex function [**8**].

It is easy to see that if $\overline{x} \in E$, and $T(\overline{x})$ is the Fréchet derivative of f at $\overline{x}$, and if $\overline{x}$ is a solution of Pre-VVIP, then $\overline{x}$ is a weak-minimum of (P).

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Hence sufficient conditions for the existence theorem of Pre-VVIP are also sufficient conditions for the existence of the weak minimum of (P). Therefore the study of Pre-VVIP is important in research concerning vector optimisation problems of η-invex functions.

In [7], F.Giannesi first introduced vector variational inequalities in a finite dimensional Euclidean space. Since then, many results have been obtained on the vector-variational inequality and vector complementary problems [**2**, **3**, 4, **13**]. In [**2**, **3**, **13**], Cheng, Yang and Cheng, considered the case $\eta(y,x) = y - x$ in Pre-VIIP and Pre-QVVIP. In [11], Parida, Sahoo and Kumar considered the case $Y = R$, $D = R_+$ and $X = R^n$ in Pre-VVIP. If $X = R^n$, $Y = R$, $D = R_+$, $\eta(y,x) = y - x$, then Pre-VVIP reduces to the well-known Hartman and Stampacchia variational inequality problem [**9**]. If $X = R^n$, $Z = R^m$, $Y = R$, $D = R_+$, $G(x) = E$ for all $x \in E$, then the Pre-Quasi VVIP reduces to the problem studied by Parida and Sen [**10**].

In this paper, we investigate existence theorems for Pre-VVIP, Pre-QVVIP and as a consequence of our results, we establish sufficient conditions for the existence theorem of a weak minima [**3**] of the problem (P).

2. Preliminaries

Throught this paper, let D^* be the polar cone of D. Let $x, y \in Y$. We denote $x \leqslant y$ if $y - x \in D$ and $x \not< y$ if $y - x \notin intD$. If D is a pointed, closed, convex cone and D induces a partial order in Y, then (Y, D) is called an ordered topological vector space.

Definition 1: Let $T : X \to L(X,Y)$, $\eta : X \times X \to X$. Then T is said to be η-monotone if $\langle T(x), \eta(x,y)\rangle - \langle T(y), \eta(x,y)\rangle \geqslant 0$ for all $x, y \in X$.

Definition 2: [**8**] Let $f : X \to Y$ be Frèchet differentiable on X. Then f is said to be η-invex on X if there exists a function $\eta : X \times X \to Y$ such that for all $x, y \in X$,

$$f(y) - f(x) \geqslant \langle Df(x), \eta(y,x)\rangle,$$

where $Df(x)$ is the Frèchet derivative of f at x.

Definition 3: Let $T : E \subseteq X \to L(X,Y)$. Then T is said to be pre-v-hemicontinuous if for all $x, y \in E$, the map $t \to \langle T(x + t(y - x)), \eta(y,x)\rangle$ is continuous at $t = 0$.

3. Main results

Lemma 1. *Let $E \subseteq X$ be a non-empty convex subset and $\eta : E \times E \to E$ be a map with $\eta(x,x) = 0$, for all $x \in E$. Suppose that $T : E \to L(X,Y)$ is η-monotone* *and pre-v-hemicontinuous and the map $\langle T(x), \eta(u,y)\rangle$ is convex with respect to $u \in E$. Then the following two problems are equivalent.*

(a) *Find $x \in E$ such that $\langle T(x), \eta(y,x)\rangle \not< 0$ for all $y \in E$.*

(b) *Find $x \in E$ such that $\langle T(y), \eta(y,x)\rangle \not< 0$ for all $y \in E$.*

PROOF: (a) That implies (b) follows immediately from the η-monotonicity of T. Conversely, if (b) holds for each $x \in E$, then

$$\langle T(\lambda y + (1-\lambda)x, \eta(\lambda y + (1-\lambda)x, x))\rangle \not< 0, \text{ for all } y \in E. \tag{1}$$

Since $\langle T(x), \eta(u,y)\rangle$ is convex with respect to u and $\eta(x,x) = 0$, it follows that

$$\langle T(x+\lambda(y-x), \eta(x+\lambda(y-x), x))\rangle \leqslant \lambda\langle T(x+\lambda(y-x), \eta(y,x))\rangle \text{ for all } 0 < \lambda < 1. \tag{2}$$

(1) and (2) imply

$$\langle T(x+\lambda(y-x), \eta(y,x))\rangle \not< 0 \text{ for all } \lambda \in (0,1). \tag{3}$$

Since T is pre-v-hemicontinuous, it follows from (3) that

$$\langle T(x), \eta(y,x)\rangle \not< 0 \text{ for all } y \in E.$$

Hence (a) is true. ∎

THEOREM 1. *Let $intD \neq \phi$ and $intD^* \neq \phi$. Let E be a nonempty, compact convex set in X, $\eta : E \times E \to E$ be a map, $\eta(x,x) = 0$, for all $x \in E$. Suppose $T : E \to L(X,Y)$ is η-monotone, pre-v-hemicontinuous and $\langle T(x), \eta(u,y)\rangle$ is convex with respect to u, and for each fixed $y \in E$, $\eta(y,x)$ is continuous with respect to x on E. Then there exists $\overline{x} \in E$ such that*

$$\langle T(\overline{x}), \eta(x,\overline{x})\rangle \not< 0 \textit{ for all } x \in E.$$

PROOF: For each fixed $y \in E$, let $F_1(y) = \{x \in E \mid \langle T(x), \eta(y,x)\rangle \not< 0\}$. Then $F_1 : E \to 2^E$. We prove that F_1 is a KKM map [**12**]. If this is not the case, there exists a finite set $A = \{x_1, \cdots, x_n\} \subseteq E$ such that $covA \not\subset \bigcup_{i=1}^{n} F_1(x_i)$, where $covA$ denotes the convex hull of A. Hence there exist $\alpha_i \geqslant 0$, for all $i = 1, \cdots, n$, $\sum_{i=1}^{n} \alpha_i = 1$ and $x = \sum_{i=1}^{n} \alpha_i x_i$ such that $x \notin \bigcup_{i=1}^{n} F_1(x_i)$. Then $x \notin F_1(x_i)$ for all $i = 1, \cdots, n$. Hence $\langle T(x), \eta(x_i, x)\rangle < 0$ for all $i = 1, \cdots, n$. Since $\eta(x,x) = 0$ and $T(x) \in L(X,Y)$, it follows that

$$0 = \langle T(x), \eta(x,x)\rangle \leqslant \sum_{i=1}^{n} \alpha_i \langle T(x), \eta(x_i, x)\rangle < 0.$$

This leads to a contradiction. Hence F_1 is a KKM map.

Let $F_2(y) = \{x \in E \mid \langle T(y), \eta(y,x)\rangle \not< 0\}$.

Since T is η-monotone, it is easy to see that F_2 is also a KKM map on E. By Lemma 1

$$\bigcap_{y\in E} F_1(y) = \bigcap_{y\in E} F_2(y).$$

Since for each fixed $y \in E$, we have $T(y) \in L(X,Y)$ and $\eta(y,x)$ is continuous with respect to $x \in E$ and $Y\backslash(-int\ D)$ is closed, it follows that $F_2(y)$ is a compact subset in E. By the F-KKM theorem [5].

$$\bigcap_{y\in E} F_1(y) = \bigcap_{y\in E} F_2(y) \neq \phi.$$

Hence there exists $\overline{x} \in E$ such that

$$\langle T(\overline{x}), \eta(x,\overline{x})\rangle \not< 0 \text{ for all } x \in E.$$

□

Lemma 2. *Let $E \subseteq X$ be a nonempty convex set and $\eta : E \times E \to E$ be a map with $\eta(x,x) = 0$ for all $x \in E$. Suppose $T = (T_1, \cdots, T_n) : E \to L(X, R^n)$ is η-monotone and pre-v-hemicontinuous. Suppose further that for fixed $x, y \in E$ and for each $i = 1, \cdots, n$, the map $\langle T_i(x), \eta(u,y)\rangle$ is strongly qasiconvex with respect to $u \in E$ and R^n is ordered by $R^n_+ = \{x = (x_1, \cdots, x_n) : x_i \geqslant 0$ for all $i = 1, \cdots, n\}$. Then the following two problems are equivalent.*

(a) *Find $x \in E$ such that $\langle T(x), \eta(y,x)\rangle \not< 0$ for all $y \in E$.*
(b) *Find $x \in E$ such that $\langle T(y), \eta(y,x)\rangle \not< 0$ for all $y \in E$.*

Proof: That (a)$\Rightarrow$ (b) is the same as Lemma 1. Conversely, suppose (b) holds. Then there exists $x \in E$ such that $\langle T(y), \eta(y,x)\rangle \not< 0$ for all $y \in E$. Let $y \in E$, $y \neq x$ and $0 < \lambda < 1$, then $\langle T(\lambda y + (1-\lambda)x), \eta(\lambda y + (1-\lambda)x, x)\rangle \not< 0$. Hence there exists $1 \leqslant i \leqslant n$ such that

$$\langle T_i(\lambda y + (1-\lambda)x, \eta(\lambda y + (1-\lambda)x, x))\rangle \geqslant 0.$$

Since $\langle T_i(x), \eta(u,y)\rangle$ is strongly quasiconvex with respect to $u \in E$,

$$\begin{aligned} 0 &\leqslant \langle T_i(\lambda y + (1-\lambda)x), \eta(\lambda y + (1-\lambda)x, x)\rangle \\ &< \max\{\langle T_i(\lambda y + (1-\lambda)x), \eta(y,x)\rangle, \langle T_i(\lambda y + (1-\lambda)x), \eta(x,x)\rangle\} \\ &= \max\{\langle T_i(\lambda y + (1-\lambda)x, \eta(y,x))\rangle, 0\}. \end{aligned}$$

Hence $\langle T_i(\lambda y+(1-\lambda)x, \eta(y,x))\rangle > 0$, and $\langle T(\lambda y+(1-\lambda)x), \eta(y,x)\rangle \not< 0$. Then following the same argument as Lemma 1, we can show that

$$\langle T(x), \eta(y,x)\rangle \not< 0 \text{ for all } y \in E.$$

□

THEOREM 2. *Let $E \subseteq X$ be a nonempty convex set in E, $\eta : E \times E \to E$ be a function, and for each fixed $y \in E$, let the map $\eta(y,x)$ be a continuous function of x on E which, $\eta(x,x) = 0$ for all $x \in E$. Suppose that $T = (T_1, \cdots, T_n) : E \to L(X, R^n)$ is η-monotone and pre-v-hemicontinuous. For fixed $x, y \in E$ and for each $i = 1, 2, \cdots, n$, suppose $\langle T_i(x), \eta(u,y)\rangle$ is strongly quasiconvex with respect to u. Suppose further that there exists a compact convex subset K of E such that for each $y \in E\backslash K$ there exists $x \in K$ with $\langle T(y), \eta(x,y)\rangle < 0$. Then there exists a $\overline{x} \in K$ such that $\langle T(\overline{x}), \eta(x,\overline{x})\rangle \not< 0$ for all $x \in E$.*

By Lemma 2 and with the same argument as in the proof of Theorem 1, we can show that for every compact set $M \subseteq E$ there exists an $\overline{x} \in M$ such that $\langle T(\overline{x}, \eta(x,\overline{x}))\rangle \not< 0$ for all $x \in M$. For each $y \in E$, let

$$K(y) = \{x \in K, \langle T(x), \eta(y,x)\rangle \not< 0\}.$$

Since $T : E \to L(X,Y)$ is continuous and $Y\backslash intD$ is a closed set, it follows that the set $K(y)$ is closed in K and hence compact. Let $\{y_1, \cdots, y_m\} \subseteq E$ and let $A = cov[K \cup \{y_1, \cdots, y_m\}]$. Thus A is a compact and convex set in E, so there exists an $\overline{x} \in E$ such that

$$\langle T(\overline{x}), \eta(y,\overline{x})\rangle \not< 0 \text{ for all } y \in A.$$

Now $\overline{x} \in K$, for otherwise, there exists a $y \in K$ such that $\langle T(\overline{x}), \eta(y,\overline{x})\rangle < 0$, which contradits (4). Since $\langle T(\overline{x}), \eta(y,\overline{x})\rangle \not< 0$ for all $x \in A$, it follows that $\overline{x} \in \bigcap_{i=1}^{m} K(y_i)$. Thus the family of closed subsets $\{K(y) : y \in E\}$ has the finite intersection property. Since K is compact, it follows that $\bigcap_{y\in E} K(y) \neq \phi$. So there exists an $x_0 \in K(y)$ for all $y \in E$. Therefore there exists a $x_0 \in K$ such that $\langle T(x_0), \eta(y,x_0)\rangle \not< 0$ for all $y \in E$.

LEMMA 3. [1] *Let $G : X \to 2^Y$ and W be a real valued function defined on $X \times Y$, $V(x) = \sup_{y\in G(x)} W(x,y)$ and $M(x) = \{y \in G(x) \mid V(x) = W(x,y)\}$. Suppose that*

(a) *W is continuous on $X \times Y$.*

(b) *G is continuous [1] with compact values [1].*

Then the set-valued map M is upper semi-continuous [1].

THEOREM 3. *Let E be a nonempty compact convex set in X and C a compact convex set in Y. Let $V : E \to 2^C$ be upper semicontinuous, convex and closed valued and let $\phi : E \times C \times E \to R$ be continuous. Suppose that*

(a) *$\phi(x,y,x) \geqslant 0$ for all $x \in E$,*

(b) *For each fixed* $(x,y) \in E \times C$, $\phi(x,y,u)$ *is quasiconvex with respect to* $u \in E$.

(c) $G : E \to 2^E$ *is continuous with compact convex values.*

Then there exists $\overline{x} \in G(\overline{x})$ *and* $\overline{y} \in V(\overline{x})$ *such that*

$$\phi(\overline{x},\overline{y},x) \geqslant 0 \text{ for all } x \in G(\overline{x}).$$

PROOF: For each $(x,y) \in E \times Y$, let

$$\pi(x,y) = \{s \in G(x) \mid \phi(x,y,s) = \min_{u \in G(x)} \phi(x,y,u)\}.$$

Then it follows from Lemma 3 that $\pi(x,y)$ is upper semicontinuous. Since $\phi(x,y,u)$ is quasiconvex with respect to u, it follows that $\pi(x,y)$ is a convex subset of E. The set-valued function $F : E \times C \to 2^E \times 2^C$ is defined by $F(x,y) = \{(\pi(x,y), V(x))\}$. Then F is nonempty, convex closed and upper semicontinuous. By the generalised Kakutani fixed point theorem [**6**], there exists $(\overline{x},\overline{y}) \in E \times C$ such that $(\overline{x},\overline{y}) \in F(\overline{x},\overline{y})$. Hence there exist a $\overline{x} \in G(\overline{x})$ and a $\overline{y} \in V(\overline{x})$ such that

$$\phi(\overline{x},\overline{y},x) \geqslant \phi(\overline{x},\overline{y},\overline{x}) \geqslant 0 \text{ for all } x \in G(\overline{x}).$$ ∎

THEOREM 4. *Let* E *be a nonempty convex set in* X *and* C *a closed convex set in* Y. *Let* $V : E \to 2^C$ *be an upper semicontinuous closed and convex valued map and let* $\phi : E \times C \times E \to R$ *be a continuous function. Suppose that*

(a) $\phi(x,y,x) \geqslant 0$ *for all* $x \in E$.

(b) *For each fixed* $(x,y) \in E \times C$, $\phi(x,y,u)$ *is quasiconvex with respect to* $u \in E$.

(c) *There exists nonempty compact convex set* $K \subseteq E$ *such that for each* $(x,y) \in E \times C$ *with* $x \notin K$, *there exists* $u \in K$ *such that* $\phi(x,y,u) < 0$.

Then there exist a $\overline{x} \in K$, *and a* $\overline{y} \in V(\overline{x})$ *such that*

$$\phi(\overline{x},\overline{y},u) \geqslant 0 \text{ for all } u \in E.$$

PROOF: Let M be a compact and convex subset of C. For each $u \in E$, let $K(u) = \{x \in K \mid \text{ there exists } y \in V(x) \cap M \text{ such that } \phi(x,y,u) \geqslant 0\}$. It is easy to see that $K(u)$ is a closed subset of K. Let $u_1, \cdots, u_m \in E$ and $W(x) = V(x) \cap M$ and $A = conv(K \cup \{u_1, \cdots, u_m\})$. Then A is a compact and convex subset of E. By Theorem 3, there exist $x_0 \in A$, $y_0 \in W(x_0) = V(x_0) \cap M$ such that $\phi(x_0,y_0,u) \geqslant 0$ for all $u \in A$. By the assumption (c), we see that $x_0 \in K$ and $x_0 \in \bigcap_{i=1}^{m} K(u_i)$. Thus the collection $\{K(u) : u \in E\}$ of closed sets in K has the finite intersection property.

We have $\bigcap_{u\in E} K(u) \neq \phi$. Hence there exists $\bar{x} \in K(u)$ for all $u \in E$. This shows that there exist $\bar{x} \in K$ and $\bar{y} \in V(\bar{x}) \cap M \subset V(\bar{x})$ such that $\phi(\bar{x},\bar{y},u) \geqslant 0$ for all $u \in E$. □

THEOREM 5. *Let E be a nonempty compact convex set in X and C be a closed convex set in Z. Let $V : E \to 2^C$ be an upper semicontinuous closed convex valued map, $H : E \times C \to L(X,Y)$ be continuous and $\eta : E \times E \to E$ be continuous functions. Suppose that*

- (a) $\eta(x,x) = 0$.
- (b) *There exists $0 \neq y^* \in D^*$ such that for each $(x,y) \in E \times C$, the function $\langle y^* \circ H(x,y), \eta(u,x)\rangle$ is quasiconvex with respect to $u \in E$.*
- (c) *$G : E \to 2^E$ is continuous with compact values.*

Then there exist $\bar{x} \in G(\bar{x})$ and $\bar{y} \in V(\bar{x})$ such that

$$\langle H(\bar{x},\bar{y}), \eta(u,\bar{x})\rangle \not< 0 \text{ for all } u \in G(\bar{x}).$$

PROOF: Let $\phi(x,y,u) = \langle y^* \circ H(x,y), \eta(u,x)\rangle$. Then the theorem follows from Theorem 3 and the assumption $0 \neq y^* \in D^*$. □

COROLLARY 1. *Let E be a nonempty compact convex set in R^n, and C be a nonempty convex set in R^m. Let $V : E \to 2^C$ be an upper semicontinuous, convex and closed valued map, let $H : E \times C \to R^n$ and $\eta : E \times E \to E$ be continuous functions. Suppose that*

- (a) $\eta(x,x) = 0$.
- (b) *For each $(x,y) \in E \times C$, the function $\langle H(x,y), \eta(u,x)\rangle$ is quasiconvex in u.*
- (c) *$G : E \to 2^E$ is continuous with compact values.*

Then there exist $\bar{x} \in G(\bar{x})$, $\bar{y} \in V(\bar{x})$ such that

$$\langle H(\bar{x},\bar{y}), \eta(u,\bar{x})\rangle \geqslant 0 \text{ for all } u \in G(\bar{x}).$$

PROOF: If we let $X = R^n$, $Y = R$, $Z = R^m$, then $H : E \times C \to L(X,Y) = L(R^n, R) = R^n$ and the Corollary follows immediately from Theorem 5. □

REMARK. If $G(x) = E$ for all $x \in E$, then Corollary 1 reduces to Theorem 2 [**11**].

THEOREM 6. *Let E be a nonempty, convex set in X, $intD = \phi$ and $intD^* \neq \phi$. Let $\eta : E \times E \to E$ be a function, $\eta(x,x) = 0$, $\eta(x,y) = -\eta(y,x)$ for all $x, y \in E$ and for each fixed $y \in E$, let $\eta(y,x)$ be continuous with respect to $x \in E$. Suppose that $f : E \to Y$ is η-invex on E with $T(x)$ be the Frèchet derivative of f at x. Suppose that T is pre-v-hemicontinuous on E and $\langle T(x), \eta(u,y)\rangle$ is convex with respect to $u \in E$. Then there exists a $\bar{x} \in E$ such that $\bar{x}$ is a weak minimum of problem (P).*

PROOF: Let $x, y \in E$. Since f is η-invex in E it is easy to see that T is η-monotone. Then by Theorem 1 and the η-invexity of f, there exists $\overline{x} \in E$ such that $\overline{x}$ is a weak minimum of (P). ∎

REFERENCES

[1] G.P. Aubin and A. Cellina, *Differential Inclusion* (Springer-Verlag, Berlin, Heidelberg, New York, 1984), pp. 51–53.

[2] G.Y. Chen, 'Existence of solutions for a vector variational inequality: An extension of the Hartmann-Stampacchia theorem', *J. Optim. Theory Appl.* **74** (1992), 445–456.

[3] G.Y. Chen and G.M. Chang, 'Vector variational inequality, and vector optimization problem', in *Lecture Notes in Economics and Mathematical Systems* **285** (Springer-Verlag, Berlin, Heidelberg, New York, 1987), pp. 408–416.

[4] G.Y. Chen and X.Q. Yang, 'The vector complementary problem and its equivalence with the weak minimal element in ordered spaces', *J. Math. Anal. Appl.* **153** (1990), 136–158.

[5] K. Fan, 'A generalization of Tychonoff's fixed point theorem', *Math. Ann.* **142** (1961), 305–310.

[6] K. Fan, 'Fixed-pointed and minimax theorems in locally convex topological linear spaces', *Proc. Nat. Acad. Sci. U.S.A.* **38** (1952), 121–126.

[7] F. Giannessi, 'Theorems of alternative, quadratic problems and complementary problems', in *Variational Inequalities and Complementary problems*, (R. W. Cottle, F. Giannessi and L. J. Lions, Editors) (Wiley, New York), pp. 151–186.

[8] M.A. Hanson, 'On the sufficiency of the Kuhn-Tucker conditions', *J. Math. Anal. Appl.* **80** (1981), 545–550.

[9] G.J. Hartman and G. Stampacchia, 'On some nonlinear differential functional equations', *Acta Math.* **115** (1966), 271–310.

[10] J. Parida and A. Sen, 'A variational-like inequality for multifunctions with applications', *J. Math. Anal. Appl.* **124** (1987), 73–81.

[11] J. Parida, M. Sahoo and A. Kumar, 'A variational-like inequality problem', *Bull. Austral. Math. Soc.* **39** (1989), 225–231.

[12] K.K. Tan, 'Comparison theorems on minimax inequalities, variational inequalities and fixed point theorems', *J. London Math. Soc.* **28** (1983), 555–562.

[13] X.Q. Yang, 'Vector complementary and minimal element problems', *J. Optim. Theory Appl.* **77** (1993), 483–495.

Department of Mathematics
National Changua University of Education
Changhua 50058
Taiwan
Republic of China