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On Some Generalized Quasi-Equilibrium Problems*

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0. INTRODUCTION

By an *equilibrium problem*, Blum and Oettli [2] understood the problem of finding

(EP) $\hat{x} \in X$ such that $f(\hat{x}, y) \leq 0$ for all $y \in X$,

where X is a given set and $f\colon X \times X \to \overline{\mathbb{R}}$ is a given function.

We can consider more general problems as follows:

A *quasi-equilibrium problem* is to find

(QEP) $\hat{x} \in X$ such that $\hat{x} \in S(\hat{x})$ and $f(\hat{x}, z) \leq 0$ for all $z \in S(\hat{x})$,

where X and f are as above and $S\colon X \multimap X$ is a given multimap.

A *generalized quasi-equilibrium problem* is to find

(GQEP) $\hat{x} \in X$ and $\hat{y} \in T(\hat{x})$ such that $\hat{x} \in S(\hat{x})$ and $f(\hat{x}, \hat{y}, z) \leq 0$ for all $z \in S(\hat{x})$,

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where X and S are the same as above, Y is another given set, $T: X \multimap Y$ is another multimap, and $f: X \times Y \times X \to \overline{\mathbb{R}}$ is a given function.

These problems contain as special cases, for instance, optimization problems, problems of the Nash type equilibrium, complementarity problems, fixed point problems, and variational inequalities, as well as many others. There are many variations or generalizations of these problems (see [3, 4, 8–13, 18]).

In this paper, we study some equilibrium problems, quasi-equilibrium problems, and generalized quasi-equilibrium problems in G-convex spaces using a new method of fixed point approach. In fact, the second author [11] obtained an existence theorem of generalized quasi-equilibrium problems in nonnecessarily locally convex spaces. On the other hand, the second author [14, 15] extended the concept of H-spaces to G-convex spaces and established the KKM theory on these spaces. By exploiting these new approaches, we obtain new theorems including the key results in [3] in more exact formulations under much weaker restrictions.

1. PRELIMINARIES

Let X and Y be nonempty sets. A *multimap* or *map* $T: X \multimap Y$ is a function from X into the power set of Y with nonempty values. Let $x \in T^{-}(y)$ if and only $y \in T(x)$.

For topological spaces X and Y, a map $T: X \multimap Y$ is said to be *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, the set $T^{-}(B) = \{x \in X: T(x) \cap B \neq \emptyset\}$ is a closed subset of X; *lower semicontinuous* (l.s.c.) if, for each open set $B \subset Y$, the set $T^{-}(B)$ is open; *continuous* if it is u.s.c. and l.s.c.; *closed* if its graph $\mathrm{Gr}(T) = \{(x, y): x \in X, y \in T(x)\}$ is closed in $X \times Y$; and *compact* if the closure $\overline{T(x)}$ of its range $T(X)$ is compact in Y.

For a set D, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D; and let Δ_n be the standard n-simplex with vertices $e_1, e_2, \ldots, e_{n+1}$, where e_i is the ith unit vector in $\mathbb{R}^{n+1}$.

Park and Kim [14, 15] introduced the concept of a *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consisting of a topological space X, a nonempty subset D of X, and a map $\Gamma: \langle D \rangle \multimap X$ such that for each $A \in \langle D \rangle$ with $|A| = n + 1$, there exists a continuous function $\phi_A: \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A \rangle$.

For a G-convex space $(X, D; \Gamma)$, a subset C of X is said to be *G-convex* if, for each $A \in \langle D \rangle$, $A \subset C$ implies $\Gamma(A) \subset C$. We may write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D \rangle$. If $D = X$, then $(X, D; \Gamma)$ will be denoted by (X, Γ).

An extended real-valued function $g: X \to \overline{\mathbb{R}}$ on a topological space X is *lower* (resp. *upper*) *semicontinuous* (l.s.c.) (resp. u.s.c.) if $\{x \in X: g(x) > r\}$ (resp. $\{x \in X: g(x) < r\}$) is open for each $r \in \overline{\mathbb{R}}$. If X is a G-convex space, then $g: X \to \overline{\mathbb{R}}$ is *G-quasi-convex* [resp. *G-quasi-concave*] if $\{x \in X: g(x) < r\}$ (resp. $\{x \in X: g(x) > r\}$) is G-convex for each $r \in \overline{\mathbb{R}}$.

It is easy to see from Park and Kim [14] that any H-space (X, D), Lassonde's convex space, and a convex subset of a topological vector space are G-convex spaces. For example, any convex space (X, D) becomes a G-convex space (X, D) by putting $\Gamma_A = \operatorname{co} A$, where $\operatorname{co} A$ denotes the convex hull of A. Throughout this paper, we assume that every space is Hausdorff, and t.v.s. means topological vector spaces.

A nonempty subset X of a t.v.s. E is said to be *admissible* (in the sense of Klee [7]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h: K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well known that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p, $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy space H^p for $0 < p < 1$, certain Orlicz spaces, and ultrabarrelled t.v.s. admitting Schauder basis. Moreover, a locally convex subset of an F-normable t.v.s. and every compact convex locally convex subset of a t.v.s. are admissible. For details, see Hadžić [5], Weber [17], and references therein.

A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. A map $T: X \multimap Y$ is said to be *acyclic* if it is u.s.c. with acyclic compact values.

The following generalized quasi-equilibrium existence theorem is needed in this paper:

THEOREM 0 [11]. *Let X and Y be admissible convex subset of t.v.s. E and F, respectively, let $S: X \multimap X$ be a compact closed map, let $T: X \multimap Y$ be a compact acyclic map, and let $\phi: X \times Y \times X \to \overline{\mathbb{R}}$ be an u.s.c. function. Suppose that*

(i) *the function $m: X \times Y \to \overline{\mathbb{R}}$ defined by*

$$m(x, y) = \max_{u \in S(x)} \phi(x, y, u) \qquad \text{for } (x, y) \in X \times Y$$

is l.s.c.; and

(ii) *for each $(x, y) \in X \times Y$, the set*

$$M(x, y) = \{u \in S(x): \phi(x, y, u) = m(x, y)\}$$

is acyclic. Then there exists an $(\bar{x}, \bar{y}) \in X \times Y$ *such that*

$$\bar{x} \in S(\bar{x}),\ y \in T(\bar{x}), \qquad \phi(\bar{x}, \bar{y}, \bar{x}) \geq \phi(\bar{x}, \bar{y}, u) \qquad \text{for all } u \in S(\bar{x}).$$

The following is well known [1]:

BERGE'S THEOREM. *Let X and Y be topological spaces,* $f\colon X \times Y \to \mathbb{R}$ *a real function,* $F\colon X \multimap Y$ *a multimap, and*

$$\hat{f}(x) = \sup_{y \in F(x)} f(x, y), \qquad G(x) = \{y \in F(x) : f(x, y) = \hat{f}(x)\} \qquad \text{for } x \in X.$$

(a) *If f is u.s.c. and F is u.s.c. with compact values, then* $\hat{f}$ *is u.s.c.*

(b) *If f is l.s.c. and F is l.s.c., then* $\hat{f}$ *is l.s.c.*

(c) *If f is continuous and F is continuous with compact values, then* $\hat{f}$ *is continuous and G is u.s.c.*

2. A SELECTION THEOREM AND THE FAN–BROWDER TYPE FIXED POINT THEOREM

We begin with the following selection theorem:

THEOREM 1. *Let X be a compact space,* (Y, Γ) *a G-convex space, and* $F\colon X \multimap Y$*. Suppose that*

(i) *for each* $x \in X$, $F(x)$ *is G-convex; and*

(ii) $X = \bigcup_{y \in Y} \operatorname{Int} F^{-}(y)$ (*that is,* F^{-} *has transfer open values; see* [3]).

Then there is a continuous function $f\colon X \to Y$ *such that* $f(x) \in F(x)$ *for all* $x \in X$*; that is, F has a continuous selection.*

Proof. Since X is compact, there exists a finite subset $B = \{y_1, y_2, \ldots, y_{n+1}\}$ of Y such that $X = \bigcup_{i=1}^{n+1} \operatorname{Int} F^{-}(y_i)$. Since (Y, Γ) is a G-convex space, there exists a continuous map $\phi_B\colon \Delta_n \to Y$ such that $\phi_B(\Delta_n) \in \Gamma_B$ and $\phi_B(\Delta_J) \subset \Gamma_J$ for each $J \in \langle B \rangle$. Let $\{\lambda_i\}_{i=1}^{n+1}$ be the partition of unity subordinated to the cover $\{\operatorname{Int} F^{-}(y_i)\}_{i=1}^{n+1}$ of X. Define a continuous map $p\colon X \to \Delta_n$ by

$$p(x) = \sum_{i=1}^{n+1} \lambda_i(x) e_i = \sum_{i \in N_x} \lambda_i(x) e_i \qquad \text{for } x \in X,$$

where $i \in N_x \Leftrightarrow \lambda_i(x) \neq 0 \Rightarrow x \in F^-(y_i) \Leftrightarrow y_i \in F(x)$. By (i), we have

$$(\phi_B p)(x) \in \phi_B(\Delta_{N_x}) \subset \Gamma_{N_x} \subset F(x).$$

Let $f := \phi_B p$. Then $f\colon X \to Y$ is continuous and $f(x) \in F(x)$ for all $x \in X$.

Note that Theorem 1 reduces to [3, Lemma 2.1(4)] whenever Y is an H-space.

By the same argument, we have the following fixed point theorem:

THEOREM 2. *Let (X, Γ) be a compact G-convex space and let $F\colon X \multimap X$ be a map such that*

(i) *for all $x \in X$, $F(x)$ is G-convex; and*

(ii) $X = \bigcup_{y \in X} \operatorname{Int} F^-(y)$.

Then F has a fixed point.

Proof. As in the proof of Theorem 1, we see that $p\phi_B\colon \Delta_n \to \Delta_n$ is continuous, hence $p\phi_B$ has a fixed point $z \in \Delta_n$; that is, $z = (p\phi_B)(z)$. Let $\bar{x} = \phi_B(z)$. Then

$$\bar{x} = \phi_B(z) = \phi_B((p\phi_B)(z)) = (\phi_B p)(\bar{x}) = f(\bar{x}) \in F(\bar{x}),$$

where $f = \phi_B p$. Therefore F has a fixed point.

As a consequence of Theorem 2, we obtain a Fan–Browder type fixed point theorem.

COROLLARY 1. *Let (X, Γ) be a compact G-convex space and let $F\colon X \multimap X$ be a map such that*

(i) *for all $x \in X$, $F(x)$ is nonempty G-convex; and*

(ii) *for all $y \in X$, $F^-(y)$ is open.*

Then F has a fixed point.

Proof. Since $F(x)$ is nonempty for all $x \in X$, there exists $y \in F(x)$. This shows that $x \in F^-(y)$ and $X = \bigcup_{y \in X} F^-(y) = \bigcup_{y \in X} \operatorname{Int} F^-(y)$.

Then all of the requirements of Theorem 2 are satisfied, and the conclusion follows.

Remark. H-space versions of Theorems 1–2 and Corollary 1 were due to Horvath in his earlier works; for the references, see [6]. Moreover, generalized forms of Theorem 2 and Corollary 1 were given by Park and Kim [14].

From Corollary 1, we have the following equilibrium existence results:

COROLLARY 2. *Let* (X, Γ) *be a compact G-convex space, and let* ψ: $X \times X \to \overline{\mathbb{R}}$ *be a function such that*

(i) *for each* $x \in X$, $\{y \in X: \psi(x, y) < 0\}$ *is G-convex*;

(ii) *for each* $y \in X$, $\{x \in X: \psi(x, y) < 0\}$ *is open*; *and*

(iii) $\psi(x, x) \geq 0$ *for all* $x \in X$.

Then there exists an $\hat{x} \in X$ *such that*

$$\psi(\hat{x}, y) \geq 0 \qquad \text{for all } y \in X.$$

Proof. Let F: $X \multimap X$ be defined by

$$F(x) = \{y \in X: \psi(x, y) < 0\} \qquad \text{for } x \in X.$$

Then

$$F^{-}(y) = \{x \in X: \psi(x, y) < 0\} \qquad \text{for } z \in X.$$

Suppose $F(x) \neq \emptyset$ for all $x \in X$. Then, by Corollary 1, there exists an $\bar{x} \in X$ such that $\bar{x} \in F(\bar{x})$, which violates condition (iii). Therefore, there exists an $\hat{x} \in X$ such that $F(\hat{x}) = \emptyset$; that is $\psi(\hat{x}, y) \geq 0$ for all $y \in X$.

COROLLARY 3. *Let* (X, Γ) *be a compact G-convex space, Y a topological space,* T: $X \multimap Y$ *a map having a continuous selection* f, *and* ϕ: $X \times Y \times X \to \overline{\mathbb{R}}$ *a function such that*

(i) $\phi(x, y, z)$ *is G-quasi-convex in* z;

(ii) $\phi(x, y, z)$ *is u.s.c. in* (x, y); *and*

(iii) $\phi(x, f(x), x) \geq 0$ *for all* $x \in X$.

Then there exist an $\hat{x} \in X$ *and a* $\hat{y} \in T(\hat{x})$ *such that*

$$\phi(\hat{x}, \hat{y}, z) \geq 0 \qquad \text{for all } z \in X.$$

Proof. Put $\psi(x, z) = \phi(x, f(x), z)$ for $(x, z) \in X \times X$. Then ψ satisfies all of the requirements of Corollary 2. Therefore, we have the conclusion.

Remark. Chang *et al.* [3, Corollary 3.2] obtained a particular form of Corollary 3 under the assumption that ϕ is continuous instead of (ii).

3. COLLECTIVELY FIXED POINT THEOREMS

We need the following version of the KKM theorem for G-KKM map:

LEMMA 1. *Let (X, Γ) be a G-convex space, let K be a nonempty compact subset of X, and $F\colon X \multimap X$ be a G-KKM map (that is, for each $N \in \langle X \rangle$, $\Gamma_N \in F(N)$) such that*

(i) $\bigcap_{x \in X} F(x) = \bigcap_{x \in X} \overline{F(x)}$ (*that is, F is transfer closed-valued; see* [3]); *and*

(ii) *for each $N \in \langle X \rangle$, there exists a compact G-convex subset L_N of X containing N such that $L_N \cap \bigcap\{\overline{F(x)}\colon x \in L_N\} \subset K$.*

Then $K \cap \bigcap_{x \in X} F(x) \neq \emptyset$.

Proof. Define $\bar{F}\colon X \multimap X$ by $\bar{F}(x) = \overline{F(x)}$ for each $x \in X$. Then $\bar{F}$ is a G-KKM map with closed values. Therefore, by Park and Kim [14, Theorem 3], we have $K \cap \bigcap_{x \in X} \overline{F(x)} \neq \emptyset$. Hence, by (i), we have the conclusion.

Note that Lemma 1 is a G-convex version of Chang *et al*. [3, Lemma 2.2] and that the coercivity condition (ii) is more general than theirs.

LEMMA 2. *Let (X, Γ) be a G-convex space, let K be a nonempty compact subset of X, let $\emptyset \neq A \subset B$ be any sets, and let $f\colon X \times X \to B$ be a function such that*

(i) *the map $y \mapsto \{x \in X\colon f(x, y) \in A\}$ is transfer closed-valued;*

(ii) *for each $x \in X$, the set $\{y \in X\colon f(x, y) \notin A\}$ is G-convex; and*

(iii) *for each $N \in \langle X \rangle$, there exists a compact G-convex subset L_N of X such that*

$$L_N \cap \bigcap_{y \in L_N} \overline{\{x \in X\colon f(x, y) \in A\}} \subset K.$$

Then one of the following holds:

(1) *there exists a $\bar{y} \in X$ such that $f(\bar{y}, \bar{y}) \notin A$; or*

(2) *there exists an $\bar{x} \in X$ such that $f(\bar{x}, y) \in A$ for all $y \in X$.*

Proof. Define $F\colon X \multimap X$ by $F(y) = \{x \in X\colon f(x, y) \in A\}$ for $y \in X$. Then (i) and (iii) imply condition (i) and (ii) in Lemma 1. Suppose that (1) does not hold. Then

$$f(y, y) \in A \qquad \text{for all } y \in X.$$

We claim that F is a G-KKM map. Otherwise, there exists an $N \in \langle X \rangle$ such that $\Gamma_N \not\subset F(N)$; that is, $z \notin F(N)$ for some $z \in \Gamma_N$. Hence $f(z, y)$

$\notin A$ for all $y \in N$ and so

$$N \subset \{ y \in X : f(z, y) \notin A \}.$$

By (ii), we have

$$z \in \Gamma_N \subset \{ y \in X : f(z, y) \notin A \}$$

and hence $f(z, z) \notin A$, which is a contradiction. Therefore F is a G-KKM map.

Now, by Lemma 1, we have

$$\bigcap_{y \in X} F(y) \neq \emptyset.$$

Therefore, there exists an $\bar{x} \in X$ such that $\bar{x} \in F(y)$ for all $y \in X$; that is, $f(\bar{x}, y) \in A$ for all $y \in X$. This completes our proof.

Note that Lemma 2 extends Chang *et al*. [3, Lemma 2.3].

From Lemmas 1 and 2, we deduce the following collectively fixed point theorem for later use:

THEOREM 3. *Let* $(X_i, \Gamma_i)_{i \in I}$ *be a family of G-convex spaces, let* $X = \prod_{i \in I} X_i$, *let K be a nonempty compact subset of X, and let* $T_i : X \multimap X_i$ *be multimaps. Suppose that, for each* $i \in I$,

(i) $T_i(x)$ *is nonempty G-convex for each* $x \in X$;

(ii) (a) $X = \bigcup_{x_i \in X_i} \operatorname{Int} T_i^-(x_i)$ *for each* $i \in I$ *whenever I is finite; or*
(b) $X = \bigcup_{y \in X} \operatorname{Int} T^-(y)$ *whenever I is infinite,*

where $T^-(y) = \bigcap_{i \in I} T_i^-(y_i)$ *for* $y = (y_i)_{i \in I} \in X$; *and*

(iii) *for each* $N \in \langle X \rangle$, *there exists a compact G-convex subset* L_N *of X containing N such that*

$$L_N \cap \bigcap_{y \in L_N} \overline{\{ x \in X : y_j \notin T_j(x) \text{ for some } j \in I \}} \subset K.$$

Then there exists an $\hat{x} \in X$ *such that*

$$\hat{x} \in \prod_{i \in I} T_i(\hat{x}) \quad \text{or} \quad \hat{x}_i \in T_i(\hat{x}) \qquad \text{for all } i \in I.$$

Proof. Suppose the contrary. Then, for any $x \in X$, we have

$$x \notin \prod_{i \in I} T_i(x). \tag{1}$$

Hence, for any $x \in X$, there exists a $j \in I$ such that $x_j \notin T_j(x)$. We define $G = \{(x, y) \in X \times X : y \notin \prod_{i \in I} T_i(x)\}$. Then, by (1), $(x, x) \in G$ for all

$x \in X$ and G is nonempty. Since $(X_i, \Gamma_i)_{i \in I}$ is a family of G-convex spaces, if we define $\Gamma\colon X \multimap X$ by $\Gamma(A) = \prod_{i \in I} \Gamma_i(\pi_i A)$ for each $A \in \langle X \rangle$, it is known that (X, Γ) is a G-convex space, where $\pi_i\colon X \multimap X_i$ is the projection for each $i \in I$ (see [16, Theorem 4.1]). By (i), for any $x \in X$, the set

$$\{y \in X\colon (x, y) \notin G\} = \left\{y \in X\colon y \in \prod_{i \in I} T_i(x)\right\} = \prod_{i \in I} T_i(x)$$

is G-convex. Moreover, for any $y \in X$, we have

$$\begin{aligned}\{x \in X\colon (x, y) \notin G\} &= \left\{x \in X\colon y \in \prod_{i \in I} T_i(x)\right\} \\ &= \left\{x \in X\colon y_i \in T_i(x) \text{ for each } i \in I\right\} \\ &= \bigcap_{i \in I} \{x \in X\colon x \in T_i^-(y_i)\} = \bigcap_{i \in I} T_i^-(y_i). \qquad (2)\end{aligned}$$

If I is infinite, by (ii)(b) and (2), the multimap $y \mapsto \{x \in X\colon (x, y) \notin G\}$ is transfer open-valued. If I is finite, by (ii)(a), T_i^- is transfer open-valued for each $i \in I$. Hence if

$$x \in \bigcap_{i \in I} T_i^-(y_i),$$

then $x \in T_i^-(y_i)$ and there exists $y_i' \in X_i$ such that

$$x \in \operatorname{Int} T_i^-(y_i') \qquad \text{for all } i \in I.$$

Since I is finite, we have

$$\begin{aligned} x \in \bigcap_{i \in I} \operatorname{Int} T_i^-(y_i') &= \operatorname{Int}\left[\bigcap_{i \in I} \operatorname{Int} T_i^-(y_i')\right] \\ &\subset \operatorname{Int}\left(\bigcap_{i \in I} T_i^-(y_i')\right) = \operatorname{Int}\{x \in X\colon (x, y') \notin G\}.\end{aligned}$$

This shows that the multimap $y \mapsto \{x \in X\colon (x, y) \notin G\}$ is transfer open-valued if I is finite.

By (iii), we see that

$$\begin{aligned} &L_N \cap \bigcap_{y \in L_N} \overline{\{x \in X\colon (x, y) \in G\}} \\ &\quad = L_N \cap \bigcap_{y \in L_N} \overline{\left\{x \in X\colon y \notin \prod_{i \in I} T_i(x)\right\}} \\ &\quad = L_N \cap \bigcap_{y \in L_N} \overline{\left\{x \in X\colon y_j \notin T_j(x) \text{ for some } j \in I\right\}} \subset K.\end{aligned}$$

Then it follows from Lemma 2 that there exists an $\bar{x} \in X$ such that $(\bar{x}, y) \in G$ for all $y \in X$; that is, $y \notin \prod_{i \in I} T_i(\bar{x})$ for all $y \in X$. This implies $\prod_{i \in I} T_i(\bar{x})$ is empty. Therefore there exists $j \in I$ such that $T_j(x)$ is empty; this contradicts condition (1). Hence there exists an $\hat{x} \in X$ such that $\hat{x} \in \prod_{i \in I} T_i(\hat{x})$.

If $(X_i, \Gamma_i)_{i \in I}$ is a family of compact G-convex spaces, then condition (iii) of Theorem 3 holds automatically. Therefore, we have the following generalization of Theorem 2.

COROLLARY 4. *Let $(X_i, \Gamma_i)_{i \in I}$ be a family of compact G-convex spaces, let $X = \prod_{i \in I} X_i$, and let T_i: $X \multimap X_i$ be a multimap for all $i \in I$. Suppose that*

(i) *for any $x \in X$ and $i \in I$, $T_i(x)$ is a nonempty G-convex set;*

(ii) *one of the following conditions holds:*

(a) *for any $i \in$, $T_i^-: X_i \multimap X$ is transfer open-valued if I is finite;*

(b) *the multimap T^-: $X \multimap X$ defined by $T^-(y) = \bigcap_{i \in I} T_i^-(y_i)$ for each $y = (y_i)_{i \in I} \in X$ is transfer open-valued if I is infinite.*

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in \prod_{i \in I} T_i(\hat{x}).$$

Remark. If I is a singleton, then Corollary 4 reduces to Theorem 2. Note that Theorem 3 and Corollary 4 are G-convex space versions of [3, Theorems 2.4 and 2.5], respectively.

4. QUASI-EQUILIBRIUM PROBLEMS

In this section, we deal with existence of solutions of certain quasi-equilibrium problems in G-convex spaces without any linear structure.

We begin with the following.

THEOREM 4. *Let (X, Γ) be a compact G-convex space, and let S: $X \multimap X$ be a map with nonempty G-convex values and open fibers (that is, $S^-(z)$ is open for each $z \in X$) such that $\bar{S}$: $X \multimap X$ is u.s.c. Suppose that ψ: $X \times X \to \overline{\mathbb{R}}$ is a continuous function such that $\psi(x, \cdot)$ is G-quasi-convex and*

$$\psi(x, x) \geq 0 \qquad \text{for all } x \in X.$$

Then there exists an $\hat{x} \in X$ such that

$$\hat{x} \in \bar{S}(\hat{x}) \quad \text{and} \quad \psi(\hat{x}, x) \geq 0 \qquad \text{for all } x \in S(\hat{x}).$$

Proof. For each natural number n, define a map $F_n\colon X \multimap X$ by

$$F_n(x) = \left\{ z \in S(x) \colon \psi(x, z) < \min_{u \in \overline{S}(x)} \psi(x, u) + \frac{1}{n} \right\}$$

for $x \in X$. Since $\psi(x, \cdot)$ is l.s.c. and $\overline{S}(x)$ is compact as closed subset of X, it attains minimum. Moreover, $\psi(x, \cdot)$ is G-quasi-convex and $S(x)$ is G-convex, and hence each $F_n(x)$ is nonempty and G-convex. Furthermore, for any $z \in X$, we have

$$F_n^-(z) = S^-(z) \cap \left\{ x \in X \colon \psi(x, z) + \max_{u \in \overline{S}(x)} [-\psi(x, u)] < \frac{1}{n} \right\}.$$

Since ψ is l.s.c. and $\overline{S}$ is u.s.c. with compact values, by Berge's theorem, $x \mapsto \max_{u \in \overline{S}(x)}[-\psi(x, u)]$ is u.s.c. Theorefore, $x \mapsto \psi(x, z) + \max_{u \in \overline{S}(x)}[-\psi(x, u)]$ is u.s.c. Note that $S^-(z)$ is open, and hence $F_n^-(z)$ is open. Therefore, by Corollary 1, F_n has a fixed point $x_n \in F_n(x_n)$ for each n. Since X is compact, we may assume $x_n \to \hat{x} \in X$. Since $x_n \in S(x_n) \subset \overline{S}(x_n)$ and the graph of $\overline{S}$ is closed, we have $\hat{x} \in \overline{S}(\hat{x})$.

On the other hand,

$$x_n \in S(x_n) \quad \text{and} \quad \psi(x_n, x_n) < \min_{u \in \overline{S}(x_n)} \psi(x_n, u) + \frac{1}{n}.$$

Since S is l.s.c. and ψ is u.s.c., the function

$$x \mapsto \inf_{u \in \overline{S}(x)} \psi(x, u) + \frac{1}{n} = - \sup_{u \in \overline{S}(x)} [-\psi(x, u)] + \frac{1}{n}$$

is u.s.c. Therefore, we have

$$\psi(\hat{x}, \hat{x}) \le \overline{\lim_{n \to \infty}} \left[\inf_{u \in \overline{S}(x_n)} \psi(x_n, u) + \frac{1}{n} \right] \le \inf_{u \in \overline{S}(\hat{x})} \psi(\hat{x}, u),$$

and hence

$$\psi(\hat{x}, x) \ge \inf_{u \in S(\hat{x})} \psi(\hat{x}, u) \ge \psi(\hat{x}, \hat{x}) \ge 0$$

for all $x \in \overline{S}(\hat{x})$ by (ii). This completes our proof.

From Theorem 4, we obtain the following GQEP, which is a correct version of Chang *et al*. [3, Theorem 3.1].

COROLLARY 5. *Let X and S be the same as in Theorem* 4, *let Y be any topological space, let $T\colon X \multimap Y$ be a map having continuous selection f:*

$X \to Y$, let and ϕ: $X \times Y \times X \to \overline{\mathbb{R}}$ be a continuous function such that

(i) *$\phi(x, y, \cdot)$ is G-quasi-convex for each $(x, y) \in X \times Y$;*

(ii) *$\phi(x, f(x), x) \geq 0$ for all $x \in X$.*

Then there exist an $\hat{x} \in \overline{S}(\hat{x})$ and a $\hat{y} = f(\hat{x}) \in T(\hat{x})$ such that

$$\phi(\hat{x}, \hat{y}, x) \geq 0 \qquad \text{for all } x \in \overline{S}(\hat{x}).$$

Proof. Put $\psi(x, z) = \phi(x, f(x), z)$ in Theorem 4.

Remarks 1. Corollary 5 improves Chang *et al.* [3, Corollary 3.3] even for convex spaces.

2. As we have seen in the above, certain GQEPs are simple consequences of corresponding QEPs whenever the map T has a continuous selection.

3. In [3, Theorem 3.1], the authors assumed that

(ii) S: $X \multimap X$ is a continuous multimap with nonempty compact H-convex values and $S^{-}(x)$ is open for any $x \in X$.

This assumption has several defects. First, since $S^{-}(z)$ is open for any $z \in X$, S is already l.s.c. Second, since S is u.s.c. with closed values and X is compact, the graph of S is closed in $X \times X$ and hence each $S^{-}(z)$ is closed for any $z \in X$. This implies that, for important examples of H-spaces such as convex spaces or contractible spaces, S becomes a constant map; that is $S(x) = X$ for all $x \in X$. Even in this case, Corollary 5 is much more general than [3, Corollary 3.2].

4. Similarly, [3, Theorems 3.4 and 3.5, Corollaries 3.5 and 3.6] could be obtained for a more simple and general situation. For example, condition (iv) of [3, Theorem 3.4] simply says that X can be covered by a finite number of sets of the form

$$\left\{x \in S^{-}(z) \colon \psi(x, f(x), z) = \min_{u \in S(x)} \psi(x, f(x), u)\right\}_{z \in X}.$$

5. GENERALIZED QUASI-EQUILIBRIUM PROBLEMS

In this section, we deduce the following GQE existence theorem from Theorem 0:

THEOREM 5. *Let X and Y be admissible convex subsets of t.v.s. E and F, respectively, let S: $X \multimap X$ be a compact continuous map with closed convex values, let T: $X \multimap Y$ be a compact acyclic map, and let ϕ: $X \times Y \times X \to \overline{\mathbb{R}}$*

be a continuous function such that

(i) $\phi(x, y, x) \geq 0$ *for all* $x \in X$ *and* $y \in T(x)$; *and*

(ii) $u \mapsto \phi(x, y, u)$ *is quasi-convex.*

Then there exist an $\bar{x} \in S(\bar{x})$ *and a* $\bar{y} \in T(\bar{x})$ *such that*

$$\phi(\bar{x}, \bar{y}, x) \geq 0 \qquad \text{for all } x \in S(\bar{x})$$

Proof. Since ϕ: $X \times Y \times X \to \overline{\mathbb{R}}$ is u.s.c. and S is l.s.c., by Berge's theorem,

$$(x, y) \to \max_{u \in S(x)} [-\phi(x, y, u)]$$

is l.s.c. on $(x, y) \in X \times Y$. Moreover, since $S(x)$ is convex and $u \mapsto \phi(x, y, u)$ is quasi-convex for each $(x, y) \in X \times Y$, the set $M(x, y) = \{u \in S(x)$: $\phi(x, Y, u) = \max_{z \in S(x)}[-\phi(x, y, z)]\}$ is convex. Therefore, by Theorem 0, there exists an $(\bar{x}, \bar{y}) \in X \times Y$ such that

$$\bar{x} \in S(\bar{x}), \quad y \in T(\bar{x}), \qquad -\phi(\bar{x}, \bar{y}, \bar{x}) \geq -\phi(\bar{x}, \bar{y}, u)$$
$$\text{for all } u \in S(\bar{x}).$$

Since $\phi(\bar{x}, \bar{y}, \bar{x}) \geq 0$ for all $\bar{x} \in X$ and $\bar{y} \in T(\bar{x})$, we have

$$\phi(\bar{x}, \bar{y}, x) \geq 0 \qquad \text{for all } x \in S(\bar{x}).$$

This completes our proof.

Remarks. 1. Note that Theorem 5 extends Chang *et al*. [3, Theorem 3.8] in several aspects.

2. As in Chen and Park [4], Theorem 5 can be applied to obtain generalized-forms of variational inequalities due to Hartman and Stampacchia, Browder, Lions and Stampacchia, Saigal, Chan and Pang, Shih and Tan, Kim, Chang, Parida and Sen, Yao, and others.

As applications of Theorem 5, we obtain the following general variational inequality:

THEOREM 6. *Let e be a real reflexive Banach space, let* $X \subset E$ *be a nonempty closed convex subset, let* Y *be an admissible convex subset of a t.v.s.* F, *and let* T: $X \multimap Y$ *be an acyclic map. Let* M: $X \times Y \to E^*$ *be a continuous map with respect to the weak topology on* X, *the topology on* Y, *and the norm topology of* E^*. *Let* η: $X \times X \to E$ *be a weakly continuous mapping (that is, a continuous mapping with respect to the weak topology on*

X and the weak topology on E). Suppose further that

(i) $\eta(x, x) = 0$ *for all* $x \in X$;

(ii) *the function* $u \mapsto \langle M(x, y), \eta(u, x)\rangle$ *is convex; and*

(iii) *there exists a* $\bar{u} \in X$ *with* $\|\bar{u}\| < r$ *such that*

$$\max_{y \in T(x)} \langle M(x, y), \eta(\bar{u}, x)\rangle \le 0 \qquad \text{for all } x \in X \text{ with } \|x\| = r.$$

Then there exist an $\bar{x} \in X$ *and a* $\bar{y} \in T(x)$ *such that*

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x})\rangle \ge 0 \qquad \text{for all } x \in X.$$

Proof. We let $\psi(x, y, u) = \langle M(x, y), \eta(u, x)\rangle$ and $X_r = X \cap B(0, r)$, where $B(0, r) = \{x \in E: \|x\| \le r\}$. Since E is a reflexive Banach space, X_r is a weakly compact convex subset of X. By assumption, we see that ψ: $X_r \times Y \times X_r \to \overline{\mathbb{R}}$ is continuous. Then it follows from Theorem 5 that there exist an $\bar{x} \in X_r$ and a $\bar{y} \in T(\bar{x})$ such that

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x})\rangle \ge 0 \qquad \text{for all } x \in X_r.$$

Then we follow the argument of Chang *et al.* [3, Theorem 3.9] and get the conclusion.

Remark. Chang *et al.* [3, Theorem 3.9] assumed that F is a Fréchet space, $Y \subset F$ is a nonempty closed convex set, and T: $X \multimap Y$ is an u.s.c. multimap with compact convex values with respect to the weak topology on X and the topology on Y.

Let E be a real t.v.s., let $X \subset E$, and let η: $X \times X \to E$ be a function with $\eta(x, x) = 0$ for all $x \in X$. A multimap G: $X \multimap E^*$ is said to be *η-monotone* [3] if for each $x, u \in X$ and any $y \in G(x)$, $v \in G(u)$, we have

$$\langle y, \eta(u, x)\rangle + \langle v, \eta(x, u)\rangle \le 0.$$

THEOREM 7. *Let E, X, F, Y, T, and M be the same as in Theorem* 6. *Let* η: $X \times X \to E$ *be a weakly continuous function such that*

(i) $\eta(x, x) = 0$ *for all* $x \in X$;

(ii) *the function* $u \to \langle M(x, y), \eta(u, x)\rangle$ *is convex and the multimap* G: $X \to 2^{E^*}$ *defined by*

$$G(x) = \{M(x, y): y \in T(x)\}$$

is η-monotone; and

(iii) *there exist a* $\bar{u} \in X$ *and a* $\bar{v} \in T(\bar{u})$ *such that*

$$\lim_{\substack{\|x\| \to \infty \\ x \in X}} \langle M(\bar{u}, \bar{v}), \eta(x, \bar{u})\rangle > 0.$$

Then there exist an $\bar{x} \in X$ *and a* $\bar{y} \in T(\bar{x})$ *such that*

$$\langle M(\bar{x}, \bar{y}), \eta(x, \bar{x}) \rangle \geq 0 \qquad \text{for all } x \in X.$$

Proof. Applying Theorem 6 and by following the argument of Chang *et al*. [3, Theorem 3.10], we can obtain the conclusion.

Remark. The results in Section 4 and 5 of [3] are consequences of our results. For example, it is easy to see that [3, Theorem 4.1] can be deduced from Theorem 2 and [3, Theorem 5.4] from Theorem 7.

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