Journal of Mathematical Analysis and Applications **233,** 581–596 (1999)
Article ID jmaa.1999.6311, available online at http://www.idealibrary.com on IDEAL®

Remarks on Fixed Points, Maximal Elements, and Equilibria of Generalized Games*

Lai-Jiu Lin

Department of Mathematics, National Changhua University of Education, Changhua, Taiwan, Republic of China 50058
E-mail: maljlin@math.ncue.edu.tw

Sehie Park

Department of Mathematics, Seoul National University, Seoul, Republic of Korea 151-742
E-mail: shpark@math.snu.edu.kr

and

Zenn-Tsuen Yu

Department of Electrical Engineering, Nan-Kai College of Technology and Commerce, Nantour, Taiwan, Republic of China 542

Submitted by Arrigo Cellina

Received February 10, 1998

1. INTRODUCTION

Recently, Mehta *et al*. [12] gave fixed points theorems for Φ-condensing multimaps in locally convex topological vector spaces, and applied them to existence theorems of maximal elements of Φ-condensing maps which are either upper semicontinuous or L_C-majorized. Moreover they deduced existence theorems of equilibria in locally convex topological vector spaces for the one-person game and generalized game which may have any (countable or uncountable) set of players.

* This research was supported by Mathematical Center of the National Science Council, Republic of China, while the second author was visiting at National Changhua University of Education. The second author was supported in part by the Non-directed Research Fund, Korea Research Foundation, 1997.

0022-247X/99 $30.00

In this paper, using a new fixed point theorem for acyclic maps due to Park [13], we obtain collectively fixed point theorems for noncompact maps defined on a noncompact convex subset of a topological vector space in a larger class than locally convex ones. Our new results are applied to existence theorems of maximal elements of Φ-condensing maps and equilibria for one-person or generalized games. We deal with multimaps defined on an admissible convex and non-compact subset of a topological vector space, not-necessarily locally convex topological vector space. Consequently our results generalize and improve the corresponding results given by Yannelis *et al.* [21], Mehta *et al.* [12], and many other results.

This paper is organized as follows:

In Sect. 2, we give some necessary terminology and facts. Our main tool is a particular form of the fixed point theorem due to Park [13, Theorem 1].

Section 3 deals with collectively fixed point theorems for later use. In Sect. 4, some of the fixed point results in the previous sections are restated in the form of existence theorems for maximal elements.

Section 5 deals with various existence theorems for equilibria of generalized games. Those results are given in the forms of full generality.

2. PRELIMINARIES

Through this paper, tvs means Hausdorff topological vector spaces. A *multimap* or a *map* $T\colon X \multimap Y$ is a function from X into the power set 2^Y of Y with values $T(x) \subset Y$ for $x \in X$ and fibers $T^-(y)$ for $y \in Y$. Note that $x \in T^-(y)$ if and only if $y \in T(x)$. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish. For topological spaces X and Y, a map $T\colon X \multimap Y$ is said to be *upper semicontinuous* (usc) if for each closed set $B \subset Y$, The set $T^-(B) = \{x \in X\colon T(x) \cap B \neq \emptyset\}$ is a closed subset of X; and *acyclic* if it is usc with acyclic compact values. A *maximal element* of $T\colon X \multimap Y$ is a point $x^* \in X$ such that $T(x^*) = \emptyset$. The *graph* $\mathrm{Gr}(T)$ of T is the set $\{(x, y) \in X \times Y\colon y \in T(x)\}$. The interior and closure are denoted by int and cl, respectively. The map $\overline{T}\colon X \multimap Y$ is defined by $\overline{T}(x) = \{y \in Y\colon (x, y) \in \mathrm{cl}_{X\times Y}\, \mathrm{Gr}(T)\}$. The map $\mathrm{cl}\, T\colon X \multimap Y$ is defined by $\mathrm{cl}\, T(x) = \mathrm{cl}(T(x))$ for each $x \in X$. For two maps $T, G\colon X \multimap Y$, the map $T \cap G\colon X \multimap Y$ is defined by $(T \cap G)(x) = T(x) \cap G(x)$ for each $x \in X$. If X is a set, Y is a subset of a vector space E and $T\colon X \multimap Y$ is a map such that $\mathrm{co}\, T(x) \subset Y$ for each $x \in X$, then the map $\mathrm{co}\, T\colon X \multimap Y$ is defined by $(\mathrm{co}\, T)(x) = \mathrm{co}\, T(x)$ for each $x \in X$.

Let X be a topological space, Y a nonempty subset of a vector space E and $\theta: X \to E$ a function. A map $\phi: X \multimap Y$ is said to be of class $\mathbb{L}_{\theta, C}$ if (a) for each $x \in X$, co $\phi(x) \subset Y$ and $\theta(x) \notin$ co $\phi(x)$; (b) there exists a map $T: X \multimap Y$ such that for each $x \in X$, $T(x) \subset \phi(x)$ and $T^{-}(y)$ is compactly open in X for each $y \in Y$; and (c) $\{x \in X: \phi(x) \neq \emptyset\} = \{x \in X: T(x) \neq \emptyset\}$; see Tan and Yuan [15] and Mehta *et al*. [12].

We now introduce a different class as follows:

A map $\phi: X \multimap Y$ is said to be of class $\mathbb{M}_\theta$ if (a) for each $x \in X$, co $\phi(x) \subset Y$ and $\theta(x) \notin$ co $\phi(x)$; (b) there exists a map $T: X \multimap Y$ such that for each $x \in X$, $T(x) \subset \phi(x)$ and $\{x \in X: T(x) \neq \emptyset\} = \bigcup\{\text{int } T^{-}(y): y \in Y\}$; and (c) $\{x \in X: \phi(x) \neq \emptyset\} = \{x \in X: T(x) \neq \emptyset\}$.

If $X = Y$ and $\theta = 1_X$, the identity map on X, we shall denote $\mathbb{L}_C$ and $\mathbb{M}$ in place of $\mathbb{L}_{\theta, C}$ and $\mathbb{M}_\theta$, respectively. Note that if X is compact, then we have $\mathbb{L}_{\theta, C} \subset \mathbb{M}_\theta$.

Let I be a (possibly infinite) set of players. Suppose that for each $i \in I$, the ith player's strategy set X_i is a nonempty subset of a tvs E_i and $X = \prod_{i \in I} X_i$ denote the product space of $\{X_i\}_{i \in I}$. If $P_i: X \multimap X_i$ is a preference map for each $i \in I$, the collection $\Gamma = (X_i, P_i)_{i \in I}$ is called a *qualitative game*. An *equilibrium point* $\hat{x} \in X$ of the game Γ is a point satisfying $P_i(\hat{x}) = \emptyset$ for all $i \in I$. A *generalized game* is a family of quadruples $\Gamma = (X_i; A_i; B_i; P_i)_{i \in I}$ such that for each $i \in I$, X_i is a nonempty subset of a tvs E_i and $A_i, B_i: X = \prod_{i \in I} X_i \multimap X_i$ are maps. An *equilibrium* of a generalized game Γ is a point $\hat{x} \in X$ such that, for each $i \in I$, $\hat{x}_i \in \overline{B}_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$.

Let E be a tvs and C a lattice with a least element 0. A function $\Phi: 2^E \to C$ is called a *measure of noncompactness* [7] if the following conditions hold for any $A, B \in 2^E$:

(1) $\Phi(A) = 0$ if and only if A is relative compact;

(2) $\Phi(\overline{\text{co}} A) = \Phi(A)$, where $\overline{\text{co}} A$ denotes the closed convex hull of A; and

(3) $\Phi(A \cup B) = \max\{\Phi(A), \Phi(B)\}$.

It follows that $A \subset B$ implies $\Phi(A) \leq \Phi(B)$. The above notion is a generalization of the set-measure γ and the ball-measure χ of noncompactness defined in terms of a family of seminorms or a norm.

Let D be a subset of E. A map $T: D \multimap E$ is said to be Φ-*condensing* if whenever $\Phi(T(\Omega)) \geq \Phi(\Omega)$ for $\Omega \in 2^D$, then Ω is relatively compact.

From now on, we assume that Φ is a measure of noncompactness on the given tvs E if necessary.

Note that each map defined on a compact set is Φ-condensing. If E is locally convex, then a compact map $T: X \multimap E$ is γ- or χ-condensing whenever $X \subset E$ is complete or E is quasi-complete.

LEMMA 1 [12]. *Let X be a nonempty closed convex subset of a* tvs E *and* $T\colon X \multimap X$ *a* Φ*-condensing map. Then there exists a nonempty compact convex subset* K *of* X *such that* $T(K) \subset K$.

In [12], E is assumed to be a locally convex tvs, but Lemma 1 is true for any tvs as we can see in its proof.

The following is well-known:

LEMMA 2 [9]. *Let X be a paracompact space Y, a convex subset of a* tvs E, *and* $S\colon X \multimap Y$ *is a map such that* $X = \bigcup\{\operatorname{int} S^{-}(y)\colon y \in Y\}$. *Then* co S *has a continuous selection; that is, there is a continuous map* $f\colon X \to Y$ *such that* $f(x) \in \operatorname{co} S(x)$ *for each* $x \in X$.

Moreover, if X itself is compact, then $f \subset \operatorname{co} A$ *for some finite subset A of Y.*

LEMMA 3 [12]. *Let X and Y be topological spaces and A a closed (resp. open) subset of X. Suppose* $F_1\colon X \multimap Y$ *and* $F_2\colon A \multimap Y$ *are lower (resp. upper) semicontinuous such that* $F_2(x) \subset F_1(x)$ *for all* $x \in A$. *Then the map* $F\colon X \multimap Y$ *defined by*

$$F(x) = \begin{cases} F_1(x) & \text{if } x \notin A; \\ F_2(x) & \text{if } x \in A \end{cases}$$

is also lower (resp. upper) semicontinuous.

A nonempty subset X of a tvs E is said to be *admissible* (in the sense of Klee [11]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h\colon K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

It is well known that every nonempty convex subset of a locally convex tvs is admissible. Other examples of admissible tvs are l^p, $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy space H^p for $0 < p < 1$ every compact convex locally convex subset of tvs, and many others. For details, see Park [13] and references therein.

We need the following particular form of Park [13, Theorem 1]:

THEOREM 0. *Let X be an admissible convex subset of a* tvs E *and* $T\colon X \multimap X$ *a compact acyclic map. Then T has a fixed point* $\bar{x} \in X$; *that is,* $\bar{x} \in T(\bar{x})$.

3. COLLECTIVELY FIXED POINTS

We begin with the following collectively fixed point theorem which is essential in this paper.

THEOREM 1. *Let I be an index set. For each $i \in I$, let X_i be a convex subset of a tvs E_i. Suppose that $X = \prod_{i \in I} X_i$ is a closed convex locally convex subset of the tvs $E = \prod_{i \in I} E_i$ and S_i, T_i: $X \multimap X_i$ are maps satisfying the following conditions*:

(1) *for each $x \in X$ and $i \in I$*, $\operatorname{co} S_i(x) \subset T_i(x)$;

(2) $X = \bigcup\{\operatorname{int} S_i^-(y)\colon y \in X_i\}$ *for each $i \in I$; and*

(3) *the map T: $X \multimap X$ defined by*

$$T(x) = \prod_{i \in I} T_i(x) \quad \textit{for } x \in X$$

is Φ-condensing.

Then there exists an $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

Proof. Since T is Φ-condensing, it follows from Lemma 1 that there is a nonempty compact convex subset K of X such that $T(K) \subset K$. Since K is compact, it follows from Lemma 2 that $\operatorname{co} S_i|_K$ has a continuous selection for each $i \in I$. Hence $T_i|_K$ has a continuous selection f_i: $K \to K_i$, where $K_i = \pi_i K$ and π_i is the projection of $E = \prod_{i \in I} E_i$ onto E_i. Let f: $K \to K$ be defined by

$$f(x) = \prod_{i \in I} f_i(x) \quad \text{for } x \in K.$$

Then f is continuous. Since K is a compact convex locally convex subset of X, K is an admissible subset of X. Then by Theorem 0, there is an $\bar{x} = (\bar{x}_i)_{i \in I} \in K$ such that $\bar{x} = f(\bar{x}) \in T(\bar{x})$. Hence for each $i \in I$, $\bar{x}_i = f_i(\bar{x}) \in T_i(\bar{x})$.

COROLLARY 1. *In Theorem* 1, *if the condition* (2) *is replaced by*

(2)′ *for each $i \in I$, $S_i^-(y)$ is compactly open in X for all $y \in X_i$ and $S_i(x) \neq \emptyset$ for all $x \in X$.*

Then there exists an $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\bar{x}_i \in T_i(\bar{x})$ for each $i \in I$.

Proof. Let S: $X \multimap X$ be defined by $S(x) = \prod_{i \in I} S_i(x)$ for $x \in X$. Since T is Φ-condensing and $S(x) \subset T(x)$ for all $x \in X$, it follows that S is also Φ-condensing. By Lemma 1, there exists a compact convex subset K of X such that $S(K) \subset K$. Let K_i be the projection of K into E_i. Then $S_i(K) \subset K_i$. Let $S_i' = S_i|_K$: $K \multimap K_i$. Since for each $y \in X_i$ and $i \in I$, $S_i^-(y)$ is compactly open in X, it follows that for each $y \in K_i$,

$$(S_i')^{-1}(y) = \{x \in K\colon y \in S_i'(x)\} = \{x \in K\colon y \in S_i(x)\} = K \cap S_i^-(y)$$

is open in K. By assumption, we see that for each $x \in K$ with $S_i'(x) \neq \emptyset$, if $y \in S_i'(x)$, then $y \in K_i$ and $x \in (S_i')^-(y)$. Consequently, $K = \bigcup\{(S_i')^-(y)\colon y \in K\} = \bigcup\{\mathrm{int}(S_i')^-(y)\colon y \in K\}$. Let $T_i' = T_i|_K$. Then for each $x \in K$ and each $i \in I$, $\mathrm{co}\, S_i'(x) \subset T_i'(x)$. Since K is a compact convex locally convex subset of X, K is an admissible closed set. Then the conclusion follows immediately from Theorem 1.

When I is a singleton, we do not need the admissibility.

THEOREM 2. *Let X be a nonempty closed convex set of a tvs E and S, T: $X \multimap X$ maps such that*

(1) *for each $x \in X$,* $\mathrm{co}\, S(x) \subset T(x)$; *and*

(2) $X = \bigcup\{\mathrm{int}\, S^-(y)\colon y \in X\}$.

If T is Φ-condensing, then T has a fixed point.

Proof. Since T is Φ-condensing, by Lemma 1, there exists a nonempty compact convex subset K of X such that $T(K) \subset K$. For any $x \in K$, there exists a $y \in S(x)$ such that $x \in S^-(y)$; that is, K is covered by $\{S^-(y)\colon y \in K\}$. Moreover, we have $K = K \cap X = \cup\{K \cap \mathrm{int}\, S^-(y)\colon y \in X\} = \bigcup\{\mathrm{int}_K S^-(y)\colon y \in X\}$. For any $x \in K$, there exists $y \in X$ such that $x \in \mathrm{int}_K S^-(y) \subset S^-(y)$, or $y \in S(x) \subset K$. Therefore, we have $K = \bigcup\{\mathrm{int}_K S^-(y)\colon y \in K\}$. By applying Lemma 2, there exists a continuous selection $f\colon K \to K$ of $T|_K$ such that $f(K) \subset \mathrm{co}\, A \subset K$ for some finite subset A of K. Then $f|_{\mathrm{co}\, A}$ has a fixed point by the Brouwer fixed point theorem, which is a particular form of Theorem 0. Therefore T has a fixed point in K. This completes our proof.

Remark. If X itself is compact, T is not-necessarily Φ-condensing in Theorem 2. This is known as the Fan–Browder fixed point theorem.

COROLLARY 2. *Let X be a nonempty closed convex subset of a tvs E. Suppose that T: $X \multimap X$ is a Φ-condensing map such that*

(1) *$T(x)$ is nonempty convex for each $x \in X$;*

(2) *for each $y \in X$, $T^-(y)$ is compactly open in X.*

Then T has a fixed point.

Remark. For a locally convex tvs E, Corollary 2 reduces to [12, Theorem 2.2]. Note also that Theorem 2 properly generalized Corollary 2.

EXAMPLE. Let $E = \mathbb{R}$, $X = (-\infty, 0]$ and $T(x) = (-\infty, x]$. Then for each $y \in X$, $T^-(y) = [y, 0]$. It is easy to see that $X = \bigcup\{\mathrm{int}_X T^-(y)\colon y \in X\}$. But for each $y \in X$, $T^-(y)$ is not compactly open.

We have one more collectively fixed point theorem:

THEOREM 3. *Let I be an index set and for each $i \in I$, X_i a convex subset of a tvs E_i. Suppose that the set $X = \prod_{i \in I} X_i$ is a convex locally convex closed subset of $E = \prod_{i \in I} E_i$ and for each $i \in I, T_i$: $X \multimap X_i$ is usc and for each $x \in X$, $T_i(x)$ is convex except for a finite number of i's for which $T_i(x)$ is acyclic. Suppose that T: $X \multimap X$ defined by $T(x) = \prod_{i \in I} T_i(x)$ for each $x \in X$ is Φ-condensing. Then there exists an $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ such that $\overline{x_i} \in T_i(\bar{x})$ for each $i \in I$.*

Proof. Since T is Φ-condensing, it follows from Lemma 1 that there is a compact convex subset K of X such that $T(K) \subset K$. By assumption, for each $i \in I$, T_i: $X \multimap X_i$ is usc. Let $T_i' = T_i|_K$ for each i, and $T' = T|_K = \prod_{i \in I} T_i'$. Since the product of a finite number of acyclic sets is acyclic, T' is usc with compact acyclic values. Thus T': $K \multimap K$ is a compact acyclic map. Since K is a compact convex locally convex subset of a tvs X, K is admissible, it follows from Theorem 0 that T' has a fixed point; that is, there exists an $\bar{x} = (\bar{x}_i)_{i \in I} \in K \subset X$ such that $\overline{x_i} \in T_i'(\bar{x}) = T_i(\bar{x})$ for all $i \in I$.

For the case I is a singleton, we have the following particular form of [13, Theorem 2]:

THEOREM 4. *Let X be a closed convex locally convex subset of a tvs E and T: $X \multimap X$ an acyclic map. If T is Φ-condensing, then T has a fixed point.*

Remarks. 1. Theorem 4 can be deduced from Theorem 0 and Lemma 1.

2. If T is convex-valued and E is locally convex, then Theorem 4 reduces to [12, Theorem 2.3]. Note that Theorem 0 and 4 include most of well-known fixed point theorems in (locally convex) topological vector spaces; see [13].

4. MAXIMAL ELEMENTS

Some of the fixed point results in the previous sections can be restated to existence theorems for maximal elements.

THEOREM 5. *Let X be a closed convex locally convex subset of a tvs E. Suppose that T: $X \multimap X$ is an usc Φ-condensing map with closed convex values. If $x \notin T(x)$ for each $x \in X$, then T has a maximal element; that is, there exists an $\bar{x} \in X$ such that $T(\bar{x}) = \emptyset$.*

Proof. Suppose that $T(x) \neq \emptyset$ for all $x \in X$. Since every nonempty convex set is acyclic, T: $X \multimap X$ becomes an acyclic map. Since T is Φ-condensing, by Theorem 4, T has a fixed point. This contradicts the hypothesis.

Remark. If E is locally convex, then Theorem 5 reduces to [12, Theorem 3.2].

THEOREM 6. *Let X be a nonempty closed convex subset of a tvs E. Suppose that T: $X \multimap X$ is a map such that*

(1) *T is Φ-condensing and $T(x)$ is convex for all $x \in X$;*

(2) *for each $y \in X$, $T^-(y)$ is compactly open; and*

(3) *for each $x \in X$, $x \notin T(x)$.*

Then T has a maximal element.

Proof. Suppose that for all $x \in X$, $T(x) \neq \varnothing$. Then by Corollary 2, there exists an $\bar{x} \in X$ such that $\bar{x} \in T(\bar{x})$. This contradicts (3). Then the conclusion follows.

Remarks. 1. In Theorem 5, if the condition (2) is replaced by $X = \bigcup\{\text{int}\, T^-(y)\colon\ y \in X\}$, then the conclusion is not true. In fact, for any $x \in X$, there exists a $y \in X$ such that $x \in \text{int}\, T^-(y) \subset T^-(y)$. Hence $y \in T(x) \neq \varnothing$.

2. If E is locally convex, then Theorem 6 reduces to [12, Theorem 3.1].

THEOREM 7. *Let X be a nonempty closed convex subset of a tvs E and T: $X \multimap X$ be of class $\mathbb{M}$. If T is Φ-condensing, then there exists a point $\bar{x} \in X$ such that $T(\bar{x}) = \varnothing$.*

Proof. Suppose that $T(x)$ is nonempty for each $x \in X$. Since T: $X \multimap X$ is of class $\mathbb{M}$, we have

(a) for each $x \in X$, $\text{co}\, T(x) \subset X$ and $x \notin \text{co}\, T(x)$;

(b) there exists a map G: $X \multimap X$ such that for each $x \in X$, $G(x) \subset T(x)$ and $\{x \in X\colon G(x) \neq \varnothing\} = \bigcup_{y \in X} \text{int}\, G^-(y)$; and

(c) $\{x \in X\colon G(x) \neq \varnothing\} = \{x \in X\colon T(x) \neq \varnothing\}$.

Since $T(x) \neq \varnothing$ for all $x \in X$, we have $X = \{x \in X\colon\ T(x) \neq \varnothing\}$. Thus $X = \bigcup\{\text{int}\, G^-(y)\colon\ y \in X\}$ by (b) and (c). Since T is Φ-condensing, it follows that $\text{co}\, T$ is Φ-condensing. By (b), we see that $\text{co}\, G(x) \subset \text{co}\, T(x)$ for all $x \in X$. Then by Theorem 2, there exists $x^* \in X$ such that $x^* \in \text{co}\, T(x^*)$. This contradicts (a). Hence there exists an $\bar{x} \in X$ such that $T(\bar{x}) = \varnothing$.

COROLLARY 3. *Let X be a nonempty closed convex subset of a tvs E and T: $X \multimap X$ be of class $\mathbb{L}_C$. If T is Φ-condensing, then there exists a point $\bar{x} \in X$ such that $T(\bar{x}) = \varnothing$.*

Proof. Suppose that $T(x)$ is nonempty for each $x \in X$. Since $T\colon X \multimap X$ is of class $\mathbb{L}_C$, it follows that

(a) for each $x \in X$, $\operatorname{co} T(x) \subset X$ and $x \notin \operatorname{co} T(x)$;

(b) there exists a map $G\colon X \multimap X$ such that for each $x \in X$, $G(x) \subset T(x)$ and $G^-(y)$ is compactly open for each $y \in X$;

(c) $\{x \in X\colon T(x) \neq \emptyset\} = \{x \in X\colon G(x) \neq \emptyset\}$.

Since T is Φ-condensing, there exists a compact convex subset K of X such that $T(K) \subset K$. Let $G' = G|_K$. Since for each $y \in K$, $G^-(y)$ is compactly open, it is easy to see $(G')^-(y)$ is open in K and $K = \bigcup\{\operatorname{int}_K (G')^-(y)\colon y \in K\}$. It is easy to check that $T|_K\colon K \multimap K$ is of class $\mathbb{M}$. Since K is a nonempty compact convex subset of E, by Theorem 7, there exists an $\bar{x} \in K$ such that $T(\bar{x}) = \emptyset$. This contradicts $T(x) \neq \emptyset$ for all $x \in X$.

Remark. If E is locally convex, then Corollary 3 reduces to [12, Theorem 3.4]. Similarly, [12, Theorem 3.5] can be improved without assuming the local convexity as follows:

COROLLARY 4. *Let X be a nonempty closed convex subset of a tvs E and $P\colon X \multimap X$ an $\mathbb{L}_C$-majorized Φ-condensing map. Then P has a maximal element.*

Remarks. 1. For the definition of $\mathbb{L}_C$-majorized map, see [12]; and the proof of [12, Theorem 3.4] works for Corollary 4.

2. In all of the results in Sects. 3 and 4, the Φ-condensing maps can be replaced by compact maps, for which the closedness of X is redundant, see Park [13]. For example, Corollary 4 can be stated for compact maps. Then we obtain a generalization of Borglin and Keiding [2, Corollary 1], Toussaint [16, Theorem 2.2], Tulcea [18, Theorem 2], and Yannelis and Prabhakas [21, Corollary 5.1].

5. EQUILIBRIA OF GENERALIZED GAMES

Applying Theorem 4 and following the argument in the proof of [12, Theorem 4.3], we have the following theorem:

THEOREM 8. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game, where I is a (countable or uncountable) set of players. Suppose that the following conditions are satisfied for each $i \in I$:*

(1) *X_i is a convex subset of a tvs E_i such that $X = \prod_{i \in I} X_i$ is a closed convex locally convex subset of the tvs $E = \prod_{i \in I} E_i$;*

(2) A_i: $X \multimap X_i$ *is an acyclic map such that for each* $x \in X$, $A_i(x)$ *is convex except a finite number of i's;*

(3) *the map* A: $X \multimap X$ *defined by* $A(x) = \prod_{i \in I} A_i(x)$ *for each* $x \in X$ *is* Φ*-condensing;*

(4) *for each* $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;

(5) *the set* $U_i = \{x \in X: A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X; *and*

(6) $A_i \cap P_i$ *is usc on* U_i *such that for each* $x \in U_i$, $(A_i \cap P_i)(x)$ *is closed convex except a finite number of i's for which* $(A_i \cap P_i)(x)$ *is acyclic.*

Then there exists an $x^* \in X$ *such that, for each* $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ *and* $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Remark. For the case E_1 is locally convex, Theorem 8 reduces to [12, Theorem 4.3]. Note that in Theorem 8, the involving maps have acyclic values on a finite number of points instead of convex values.

The following is a variant of [12, Theorem 4.1]:

THEOREM 9. *Let* X *be a nonempty closed convex subset of tvs* E. *Suppose* A, B, P: $X \multimap X$ *are maps such that*

(1) *for each* $x \in X$, $A(x)$ *is nonempty and* co $A(x) \subset B(x)$;

(2) *for each* $y \in X$, $A^{-}(y)$ *is compactly open in* X;

(3) $A \cap P$ *is of class* $\mathbb{M}$; *and*

(4) *the map* A *is* Φ*-condensing.*

Then there exists a point $x^* \in X$ *such that* $x^* \in \overline{B}(x^*)$ *and* $P(x^*) \cap B(x^*) = \emptyset$.

Proof. Since A is Φ-condensing, by Lemma, 1 there exists a compact convex subset K of X such that $A(K) \subset K$. Let $M = \{x \in K: x \neq \overline{B}(x)\}$. Then M is open in K. Define S: $K \multimap K$ by

$$S(x) = \begin{cases} A'(x) \cap P(x) & \text{if } x \in K \setminus M, \\ A'(x) & \text{if } x \in M, \end{cases}$$

where $A' = A|_K$.

Since $A \cap P$ is of class $\mathbb{M}$, for each $x \in X$, $x \notin \operatorname{co}(A(x) \cap P(x))$ and there exists a map β: $X \multimap X$ such that

(a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$;

(b) $\{x \in X: \beta(x) \neq \emptyset\} = \{x \in X: A(x) \cap P(x) \neq \emptyset\}$; and

(c) $\{x \in X: \beta(x) \neq \emptyset\} = \bigcup\{\operatorname{int} \beta^{-}(y): y \in Y\}$.

Now we define $T\colon X \multimap X$ by

$$T(x) = \begin{cases} \beta'(x) & \text{if } x \in K \setminus M, \\ A'(x) & \text{if } x \in M, \end{cases}$$

where $\beta' = \beta|_K$.

We want to show that $\{x \in K\colon \beta'(x) \neq \emptyset\} \subset \bigcup_{y \in K} \operatorname{int}_K(\beta')^-(y)$. In fact,

$$\begin{aligned} \{x \in K\colon \beta'(x) \neq \emptyset\} &= \{x \in K\colon \beta(x) \neq \emptyset\} = K \cap \{x \in X\colon \beta(x) \neq \emptyset\} \\ &= K \cap \left(\textstyle\bigcup_{y \in X} \operatorname{int} \beta^-(y)\right) \\ &= \bigcup\{K \cap \operatorname{int} \beta^-(y)\colon y \in X\} \\ &\subset \bigcup\left\{\operatorname{int}_K(\beta')^-(y)\colon y \in X\right\}. \end{aligned}$$

It is easy to see that for each $y \in K$, $T^-(y) = [M \cup (\beta')^-(y)] \cap (A')^-(y)$ and $(A')^-(y) = K \cap A^-(y)$ is open in K. Let $x \in \{u \in K\colon T(u) \neq \emptyset\}$. Then either $x \in M$ or $x \in K \setminus M$.

Case 1. If $x \in M$, then $T(x) = A'(x) \neq \emptyset$. Let $y \in A'(x) \subset K$, then $x \in (A')^-(y)$. Hence $x \in [M \cup \operatorname{int}(\beta')^-(y)] \cap (A')^-(y) \subset T^-(y)$. But $[M \cup \operatorname{int}(\beta')^-(y)] \cap (A')^-(y)$ is open in K, hence $x \in [M \cup \operatorname{int}(\beta')^-(y)] \cap (A')^-(y) \subset \operatorname{int}_K T^-(y) \subset \bigcup\{\operatorname{int}_K T^-(y)\colon y \in K\}$.

Case 2. If $x \in K \setminus M$, then $T(x) = \beta'(x) \neq \emptyset$. Thus $x \in \{u \in K\colon \beta'(x) \neq \emptyset\} \subset \bigcup\{\operatorname{int}_K \beta'^-(y)\colon y \in X\}$. There exists $y \in X$ such that $x \in \operatorname{int}(\beta')^-(y)$. This shows that $y \in \beta'(x) \subseteq A(x) \cap P(x) \subset A(x) = A'(x) \subset K$. Hence $x \in (A')^-(y)$ and $x \in [M \cup \operatorname{int}(\beta')^-(y)] \cap (A')^-(y) \subset \bigcup\{\operatorname{int}_K T^-(y)\colon y \in K\}$.

In any case, we see that $\{x \in K\colon T(x) \neq \emptyset\} \subset \bigcup\{\operatorname{int}_K T^-(y)\colon y \in K\}$. But $\bigcup\{\operatorname{int}_K T^-(y)\colon y \in K\} \subset \{x \in K\colon T(x) \neq \emptyset\}$. Hence $\{x \in K\colon T(x) \neq \emptyset) = \bigcup\{\operatorname{int}_K T^-(y)\colon y \in K\}$. Then clearly for each $x \in X$, $T(x) \subset S(x)$ and $\{x \in X\colon S(x) \neq \emptyset\} = \{x \in X\colon T(x) \neq \emptyset\}$. Moreover, if $x \in M$, then $x \notin \overline{B}(x)$, it follows from (1) that $x \notin \operatorname{co} A(x) = \operatorname{co} S(x)$, if $x \notin M$, then $x \notin \operatorname{co}(A(x) \cap P(x)) = \operatorname{co} S(x)$. This shows that S is of class $\mathbb{M}_{I_K}$. Since A is Φ-condensing and $S(x) \subset A(x)$ for all $x \in X$, S is Φ-condensing. Then by Theorem 2, there exists $\bar{x} \in K$ such that $S(\bar{x}) = \emptyset$. As $A(x) \neq \emptyset$ for all $x \in X$, we must have $\bar{x} \notin M$ and hence $\bar{x} \in \overline{B}(\bar{x})$ and $A(\bar{x}) \cap P(\bar{x}) = \emptyset$. Now the proof is completed.

COROLLARY 5. *Let X be a nonempty closed convex subset of a tvs E. Suppose $A, B, P\colon X \multimap X$ are maps such that*

(1) *for each $x \in X$, $A(x)$ is nonempty and* $\operatorname{co} A(x) \subset B(x)$;

(2) *for each $y \in Y$, $A^{-}(y)$ is compactly open in X;*

(3) *$A \cap P$ is of class $\mathbb{L}_C$;*

(4) *the map A is Φ-condensing.*

Then there exists a point $\bar{x} \in X$ such that $\bar{x} \in \overline{B}(\bar{x})$ and $A(\bar{x}) \cap P(\bar{x}) = \varnothing$.

Proof. Since A is Φ-condensing, by Lemma 1 there exists a compact convex subset K of X such that $A(K) \subset K$. By assumption, $A \cap P$ is of class $\mathbb{L}_C$. Hence for all $x \in X$, $x \notin \operatorname{co}(A(x) \cap P(x))$ and there exists a map $\beta\colon X \multimap X$ such that

(a) for each $x \in X$, $\beta(x) \subset A(x) \cap P(x)$,

(b) for each $y \in X$, $\beta^{-}(y)$ is compactly open,

(c) $\{x \in X\colon \beta(x) \neq \varnothing\} = \{x \in X\colon A(x) \cap P(x) \neq \varnothing\}$. Let $\beta' = \beta|_K$.

Then $\beta'\colon K \multimap K$ and for all $y \in K$, $(\beta')^{-}(y)$ is open in K. Therefore $\{x \in K\colon \beta'(x) \neq \phi\} = \bigcup\{\operatorname{int}_K(\beta')^{-}(y)\colon y \in K\}$. Let $A' = A|_K$, $B' = B|_K$ and $P' = P|_K$. Then it is easy to see that $A' \cap P'$ is of class $\mathbb{M}_{I_K}$. Then the conclusion follows from Theorem 9.

Remark. In case E is locally convex, Corollary 5 reduces to Mehta *et al*. [12, Theorem 4.1].

From now on, we give various solutions of equilibria problems for generalized games:

THEOREM 10. *Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game, where I is a set of players. Suppose that the following conditions are satisfied for each $i \in I$:*

(1) *X_i is a convex subset of a tvs E_i and $X = \prod_{i \in I} X_i$ is a closed convex locally convex subset of $E = \prod_{i \in I} E_i$;*

(2) *for each $x \in X$, $A\colon X \multimap X_i$ is usc with nonempty compact convex values;*

(3) *the mapping $A\colon X \multimap X$ defined by $A = \prod_{i \in I} A_i$ for each $x \in X$ is Φ-condensing;*

(4) *the set $U_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \varnothing\}$ is paracompact and open in X;*

(5) *$U_i = \bigcup\{\operatorname{int} T_i^{-}(y)\colon y \in X_i\}$ and for each $x \in U_i$, $T_i(x)$ is convex, where $T_i = A_i \cap P_i$.*

Then there exists an $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \varnothing$.

Proof. For each $i \in I$, by (4), (5) and Lemma 2, there exists a continuous function $f_i: U_i \to X_i$ such that $f_i(x) \in A_i(x) \cap P_i(x)$ for each $x \in U_i$. Now we define $F_i: X \multimap X_i$ by

$$F_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i \\ A_i(x) & \text{if } x \notin U_i. \end{cases}$$

Then (2), (4) and Lemma 3 imply that F_i is usc with nonempty compact convex values. Let $F: X \multimap X$ be a mapping defined by $F(x) = \prod_{i \in I} F_i(x)$. Then $F: X \multimap X$ is usc and Φ-condensing and $F(x)$ is closed and convex for each $x \in X$. Now, by applying Theorem 4, we have the conclusion.

THEOREM 11. *Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be a generalized economy, where I is the set of agents such that for each $i \in I$,*

(1) *X_i is a convex subset of a tvs E_i;*

(2) *$X = \prod_{i \in I} X_i$ is a closed convex locally convex subset of $E = \prod_{i \in I} E_i$;*

(3) *for each $x \in X$, $B_i(x)$ is nonempty convex and $A_i(x) \subset B_i(x) \subset X_i$;*

(4) *the mapping* cl B_i: $X \multimap X_i$ *defined by* $(\text{cl } B_i)(x) = \text{cl}(B_i(x))$ *for each $x \in X$ is usc;*

(5) *the map* cl B: $X \multimap X$ *defined by* $(\text{cl } B)(x) = \prod_{i \in I} \text{cl } B_i(x)$ *for each $x \in X$ is Φ-condensing;*

(6) *the set $W_i = \{x \in X: A_i(x) \cap P_i(x) \neq \emptyset\}$ is a paracompact open subset of X;*

(7) *for each $x = (x_i)_{i \in I} \in X$, $x_i \notin \text{co}[A_i(x) \cap P_i(x)]$ and*

(8) *the map $T_i: X \multimap X_i$ defined by*

$$T_i(x) = A_i(x) \cap P_i(x) \quad \textit{for } x \in X$$

has the property $W_i = \bigcup\{\text{int } T_i^-(y): y \in X_i\}$.

Then Γ has an equilibrium point $\bar{x} = (\bar{x}_i)_{i \in I} \in X$; that is, for each $i \in I$, $\overline{x_i} \in \text{cl } B_i(\bar{x})$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

Proof. For each $i \in I$, let $S_i: X \multimap X_i$ be defined by $S_i(x) = \text{co } T_i(x)$. By Lemma 2, for each $i \in I$, $S_i|_{W_i}$ has a continuous selection $f_i: W_i \to X_i$. Define

$$G_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in W_i \\ \text{cl } B_i(x) & \text{if } x \notin W_i. \end{cases}$$

Then it follows from Lemma 3 that G_i is usc. Let $G: X \multimap X$ be defined by $G(x) = \prod_{i \in I} G_i(x)$ for $x \in X$. Then G is usc with nonempty closed convex values. Since B is Φ-condensing and $G(x) \subset B(x)$ for all $x \in X$, it is easy to see that G is Φ-condensing. By Theorem 4, there exists $\bar{x} \in X$ such that $\bar{x} = (\bar{x}_i)_{i \in I} \in G(\bar{x})$; that is, $\overline{x_i} \in G_i(\bar{x})$ for each $i \in I$. Hence $\overline{x_i} \in \text{cl } B_i(x)$ and $A_i(\bar{x}) \cap P_i(\bar{x}) = \emptyset$.

Remark. A compact map version of Theorem 11 for locally convex tvs was due to Wu and Shen [20, Theorem 10], which includes earlier works of Yannelis and Prabhakar [21, Theorem 6.1], Chang [3, Theorem 3.1], and Tian [19].

COROLLARY 6. *In Theorem* 11, *if the condition* $W_i = \bigcup\{\text{int } T_i^-(y)\colon y \in X_i\}$ *is replaced by the condition that for each* $y \in X_i$, $T_i^-(y)$ *is compactly open. Then we have the same conclusion.*

Proof. Since cl $B\colon X \multimap X$ is Φ-condensing, it follows that there exists a compact convex subset K of X such that cl $B\colon K \multimap K$. By assumption, $T_i(x) = A_i(x) \cap P_i(x) \subset B_i(x)$ for all $x \in K$. Hence $T_i\colon K \multimap K_i$. Let $T_i' = T_i|_K$. Since for each $y \in X_i$, $T_i^-(y)$ is compactly open. It follows that $T_i^-(y)$ is open for each $y \in K_i$. Let $W_i' = W_i \cap K$. Then W_i' is paracompact. It is easy to see that $W_i' = \bigcup\{\text{int}_K T_i^-(y)\colon y \in K_i\}$. Then the conclusion follows from Theorem 11.

THEOREM 12. *Let I be any set of players. For each* $i \in I$, *suppose that* X_i *is a compact convex subset of a tvs* E_i, $X = \prod_{i \in I} X_i$ *is an admissible subset of* $E = \prod_{i \in I} E_i$ *and* $P_I\colon X \multimap X_i$. *Let* $U_i = \{x \in X\colon P_i(x) \neq \emptyset\}$ *be paracompact and open for each* $i \in I$ *and* $P_i(x)$ *is convex for all* $x \in U_i$ *and* $i \in I$. *Suppose further that* $U_i = \bigcup\{\text{int } P_i^-(y)\colon y \in X_i\}$, *then there exists an* $x^* \in X$ *such that for each* $i \in I$, *either* $\pi_i(x^*) \in P_i(x^*)$ *or* $P_i(x^*) = \emptyset$.

Proof. It follows from Lemma 2 that $P_i|_{U_i}\colon U_i \multimap X_i$ has a continuous selection $f_i\colon U_i \to X_i$. Define $F_i\colon X \multimap X_i$ by

$$F_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i; \\ X_i & \text{if } x \notin U_i, \end{cases}$$

It follows from Lemma 3 that F_i is usc with nonempty compact convex values such that

$$F_i(x) = \{f_i(x)\} \subset P_i(x) \quad \text{for all } x \in U_i.$$

Now define $F\colon X \multimap X$ by $F(x) = \prod_{i \in I} F_i(x)$ for each $x \in X$. Then F is usc with nonempty compact convex values. Since X is an admissible compact set and $F\colon X \multimap X$ is a compact acyclic map, it follows from Theorem 0 that there exists $x^* \in X$ such that $x^* \in F(x^*)$. It follows that for each $i \in I$, either $\pi_i(x^*) \in P_i(x^*)$ or $P_i(x^*) = \emptyset$.

THEOREM 13. *In Theorem* 12, *if we assume that* X_i *is a nonempty compact convex subset of a finite dimensional space* E_i *and* U_i *is closed*, *then we have the same conclusion*.

Proof. Note that $X = \prod_{i \in I} X_i$ is automatically admissible. It follows from Lemma 2 that $P_i|_{U_i}$ has a continuous selection $f_i: U_i \to X_i$. Define

$$G_i(x) = \begin{cases} \{f_i(x)\} & \text{if } x \in U_i; \\ X_i & \text{if } x \notin U_i. \end{cases}$$

Then, by following the proof of [12, Theorem 5.3], we have the conclusion.

Remark. Some results related to this paper can be found in [1, 4–6, 8, 10, 14, 17].

REFERENCES

1. J. P. Aubin and A. Cellina, ‘‘Differential Inclusion,’’ Springer-Verlag, Berlin–Heidelberg, 1984.
2. A. Borglin and H. Keiding, Existence of equilibrium actions and of equilibrium: A note on the new existence theorems, *J. Math. Econom.* **3** (1976), 313–316.
3. S. Y. Chang, On the nash equilibrium, *Soochow J. Math.* **16** (1990), 241–248.
4. X. P. Ding, W. K. Kim, and K. K. Tan, Equilibria of non-compact generalized games with L^*-majorized preference correspondences, *J. Math. Anal. Appl.* **164** (1992), 508–517.
5. X. P. Ding and K. K. Tan, A minimax inequality with applications to existence of equilibrium point and fixed point, *Colloq. Math.* **63** (1992), 233–237.
6. X. P. Ding and K. K. Tan, On equilibria of non-compact generalized games, *J. Math. Anal. Appl.* **177** (1993), 226–238.
7. P. M. Fitzpatrick and P. M. Petryshyn, Fixed point theorems for multivalued noncompact acyclic mappings, *Pacific J. Math.* **54** (1974), 17–23.
8. D. Gale and A. Mas-Colell, On the role of complete transitive preferences in equilibrium theory, in ‘‘Equilibrium and Disequilibrium in Economic Theory’’ (G. Schwodiauer, Ed.), pp. 7–14, Reidel, Dordrecht, The Netherlands, 1978.
9. C. D. Horvath, Contractibility and generalized convexity, *J. Math. Anal. Appl.* **156** (1991), 341–357.
10. S. M. Im, W. K. Kim, and D. I. Rim, Existence of equilibrium in non-compact sets and its applications, *J. Korean Math. Soc.* **29** (1992), 361–373.
11. V. Klee, Leray-Schauder theory without local convexity, *Math. Ann.* **141** (1960), 286–297.
12. G. Mehta, K. T. Tan and X. Z. Yuan, Fixed points, maximal elements and equilibria of generalized games, *Nonlinear Analysis, Theory, Methods and Applications,* **28** (1997), 689–699.
13. S. Park, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* **35** (1998), 803–829.
14. W. Shafer and H. Sonnenschein, Equilibrium in abstract economics without ordered preferences, *J. Math. Econom.* **2** (1975), 345–348.
15. K. K. Tan and X. Z. Yuan, Maximal elements and equilibria for μ-majoried preferences, *Bull. Austral. Math. Soc.* **49** (1994), 47–54.

16. S. Toussaint, On the existence of equilibria in economics with infinite commodities and without ordered preferences, *J. Econom. Theory* **33** (1984), 98–115.
17. C. I. Tulcea, ''On the equilibriums of generalized games,'' The center for Math. Studies in Economic and Management Science, paper No. 696, 1986.
18. C. I. Tulcea, On the approximation of upper semi-continuous correspondences and the equilibriums of generalized games, *J. Math. Anal. Appl.* **136** (1988), 267–289.
19. G. Tian, Equilibrium in abstract economies with a non-compact infinite number of agents and without ordered preferences, *Econom. Lett.* **33** (1990), 203–245.
20. X. Wu and S. Shen, A further generalization of Yannelis-Prabhakar's continuous selection theorem and its applications, *J. Math. Anal. Appl.* **197** (1996), 61–74.
21. N. C. Yannelis and N. Prabhakar, Existence of maximal elements and equilibria in linear topological spaces, *J. Math. Econom.* **12** (1983), 233–245.