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A KKM TYPE THEOREM AND ITS APPLICATIONS

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In this paper we establish a generalised KKM theorem from which many well-known *KKM* theorems and a fixed point theorem of Tarafdar are extended.

1. INTRODUCTION

In [**6**], Knaster, Kuratoaski and Mazurkiewicz established the well known *KKM* theorem on the closed cover of a simplex. In [**4**], Ky Fan generalised the *KKM* theorem to a subset of any topological vector space. There are many generalisations and many applications of this theorem.

In this paper, we establish a generalised *KKM* theorem on a generalised convex space as follows:

THEOREM 1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space and $T \in$ G-KKM(X, Y) be compact, and $G : D \to 2^Y$. Suppose that*

(1.1) *for each $x \in D$, Gx is compactly closed in Y; and*

(1.2) *for any $N \in \langle D \rangle$, $T(\Gamma_N) \subseteq G(N)$.*

Then $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

Applying Theorem 1, we extend many well-known generalised *KKM* theorem, and we give a unified treatment of these theorems (see [**5, 7, 9, 10, 12, 14, 15, 16**]). We also obtain some equivalent forms of Theorem 1 and extend a fixed point theorem of Tarafdar [**15**].

2. PRELIMINARIES

Let X, Y and Z be nonempty sets; 2^Y will denote the power set of Y. Let $F : X \to 2^Y$ be a set-valued map, $A \subseteq X$, $B \subseteq Y$ and $y \in Y$. We define

$$F^-(B) = \big\{x \in X : F(x) \cap B \neq \emptyset\big\}, \qquad F^-(y) = \big\{x \in X : y \in F(x)\big\},$$
$$F(A) = \bigcup\big\{F(x) : x \in A\big\}, \qquad G_r(F) = \big\{(x, y) : y \in F(x), x \in X\big\}.$$

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For topological spaces X and Y, a map $F : X \to 2^Y$ is said to be upper semicontinuous if the set $F^-(A)$ is closed in X for each closed subset A of Y. F is said to be closed if $G_r(F)$ is a closed subset of $X \times Y$, and F is said to be compact if $\overline{F(X)}$ is a compact subset of Y. A subset B of Y is said to be compactly closed (compactly open) if for each compact subset K of Y, the set $B \cap K$ is closed (open) in K.

Given two set-valued maps $F : X \to 2^Y$, $G : Y \to 2^Z$ the composite $GF : X \to 2^Z$ is defined by $GF(x) = G\big(F(x)\big)$ for $x \in X$. Let $\mathbb{X}$ be a class of set-valued maps. We write $\mathbb{X}(X,Y) = \{T : X \to 2^Y \mid T \in \mathbb{X}\}$, $\mathbb{X}_c(X,Y) = \{T_nT_{n-1}\cdots T_1 : T_i \in \mathbb{X},\ i = 1,2,\ldots,n$ for some $n\}$, that is, the set of finite composites of maps in X.

The following notion of an abstract class of set-valued maps was introduced by Park [**10**]. A class U of set-valued maps is one satisfying the following:

(i) U contains the class $\mathbb{C}$ of single-valued continuous functions;

(ii) each $T \in U_c$ is upper semicontinuous with compact values; and

(iii) for each polytope P, each $T \in U_c(P,P)$ has a fixed point.

We write $U_c^\kappa(X,Y) = \{T : X \to 2^Y |$ for any compact subset K of X, there is $F \in U_c(K,Y)$ such that $F(x) \subseteq T(x)$ for each $x \in K\}$. Each $F \in U_c^\kappa$ is said to be admissible.

Let X be a convex set in a vector space and D a nonempty subset of X. Then (X,D) is called a convex space if the convex hull of any nonempty finite subset of D is contained in X and X has the topology that induces the Euclidean topology on the convex hull of its finite subsets. For a nonempty subset D of X, let $\langle D\rangle$ denote the set of all nonempty finite subsets of D. Let Δ_n denote the standard n-simplex with vertices $e_1, e_2, \ldots, e_{n+1}$, where e_i is the ith unit vector in $\mathcal{R}^{n+1}$, that is $\Delta_n = \Big\{u \in \mathcal{R}^{n+1} : u = \sum_{i=1}^{n+1} \lambda_i(u)e_i, \lambda_i(u) \geqslant 0, \sum_{i=1}^{n+1} \lambda_i(u) = 1\Big\}$.

A generalised convex space [**12**] or a G-convex space $(X,D;\Gamma)$ consists of a topological space X, a nonempty subset D of X and a function $\Gamma : \langle D\rangle \to 2^X$ with nonempty values such that

1. for each $A, B \in \langle D\rangle$, $A \subset B$ implies $\Gamma(A) \subseteq \Gamma(B)$ and
2. for each $A \in \langle D\rangle$, with $|A| = n+1$, there exists a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A\rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$, where Δ_J denotes the face of Δ_n corresponding to $J \in \langle A\rangle$.

We see from [**12**] that a convex subset of a topological vector space, Lassonde's convex space, S-contractible space, H-space, a metric space with Michael's convex structure, Komiya's convex space, Bielawski's simplicial convexity, Joo's pseudoconex space are examples of G-convex spaces.

For a G-convex space $(X,D;\Gamma)$, a subset C of X is said to be G-convex if for each $A \in \langle D\rangle$, $A \subseteq C$ implies $\Gamma(A) \subset C$. We sometimes write $\Gamma(A) = \Gamma_A$ for each $A \in \langle D\rangle$.

DEFINTION 1: Let $(X, D; \Gamma)$ be a G-convex space, $T : X \to 2^Y$ and $S : D \to 2^Y$ be two set-valued maps such that $T(\Gamma_A) \subset S(A)$ for each $A \in \langle D \rangle$. Then we call S a generalised G-KKM map with respect to T. Let $T : X \to 2^Y$ be a set-valued map. T is said to have the G-KKM property if whenever $S : D \to 2^Y$ is any generalised G-KKM map with respect to T, then the family $\{\overline{Sx} : x \in D\}$ has the finite intersection property. We let $G\text{-}KKM\ (X,Y) = \{T : X \to 2^Y \mid T \text{ has the } G\text{-}KKM \text{ property}\}$. If (X, D) is a convex space, and $\Gamma_A = \operatorname{Co} A$ is the convex hull of A, then $G\text{-}KKM(X,Y) = KKM\ (X,Y)$ as defined in [**3**].

LEMMA 1. *Let $(X, D; \Gamma)$ be a G-convex space, and Y a Hausdorff space. Then $U_c^\kappa(X,Y) \subseteq G\text{-}KKM(X,Y)$*

PROOF: Lemma 1 follows immediately from the corollary of [**13**, Theorem 2] and Definition 1. ∎

LEMMA 2. [**1**] *Let Y be a compact space and $F : X \to 2^Y$ be closed. Then F is upper semicontinuous.*

LEMMA 3. [**1**] *Let $F : X \to 2^Y$ be upper semicontinuous with compact values from a compact space X to Y. Then $F(X)$ is compact.*

LEMMA 4. [**1**] *Let $X \to 2^Y$ be upper semicontinuous with closed values. Then F is closed.*

LEMMA 5. [**3**] *Let X be a convex subset of a linear space, and Y be a topological space. Then $T \in KKM(X,Y)$ if and only if $T|_P \in KKM(P,Y)$ for each polytope P in X.*

LEMMA 6. *Let X be a convex subset of a linear space, Y a topological space, A a convex subset of X, and $T \in KKM(X,Y)$. Then $T|_A \in KKM(A,Y)$.*

PROOF: Let P be any polytope in A. Since $T \in KKM(X,Y)$, it follows from Lemma 5 that $T|_P \in KKM(P,Y)$. But $(T|_A)|_P = T|_P \in KKM(P,Y)$. Again by applying Lemma 5, $T|_A \in KKM(A,Y)$. ∎

A nonempty topological space is acyclic if all its reduced Cěch homology groups over rationals vanish. In particular, any contractible space is acyclic, any convex or star-shaped space is acyclic. For a convex space Y, $k(Y)$ denotes the set of all nonempty compact convex subsets of Y, $ka(Y)$ denotes the set of all compact acyclic subsets of Y and $V(X,Y) = \big\{T \mid T : X \to ka(Y) \text{ is upper semicontinuous}\big\}$. Throughout this paper, all topological spaces are assumed to be Hausdorff.

3. MAIN RESULTS

We prove a generalised G-KKM theorem which gives a unified approach to KKM-type theorems.

THEOREM 1. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space and $T \in$ G-KKM(X,Y) be compact, $G : D \to 2^Y$. Suppose that*

(1.1) *for each $x \in D$, Gx is compactly closed in Y; and*

(1.2) *for any $N \in \langle D\rangle$, $T(\Gamma_N) \subseteq G(N)$.*

Then $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

PROOF: Since T is compact, there exists a compact set K of Y such that $T(X) \subseteq K$. From this, we see that $\overline{T(X)}$ is compact. For each $x \in D$, let $Sx = \overline{T(X)} \cap Gx$, then it follows from (1.1) that Sx is closed in $\overline{T(X)}$ for each $x \in D$. By (1.2), we see that for any $N \in \langle D\rangle$, $T(\Gamma_N) = T(\Gamma_N) \cap \overline{T(X)} \subseteq G(N) \cap \overline{T(X)} = S(N)$. Hence S is G-KKM with respect to T. It follows that $\{Sx : x \in D\} = \{\overline{Sx} : x \in D\}$ has the finite intersection property. Since $\overline{T(X)}$ is compact and $\{Sx : x \in D\}$ is a family of closed subsets in $\overline{T(X)}$, we have $\bigcap\{Sx : x \in D\} \neq \emptyset$. Therefore $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$. □

REMARK 1. In Theorem 1, if the condition $T \in G$-KKM (X,Y) is compact is replaced by the condition that $T \in U_c^\kappa(X,Y)$ and X is compact, then we obtain the following corollary.

COROLLARY 1. *Let $(X, D; \Gamma)$ be a compact G-convex space, Y a Hausdorff space, and $T \in U_c^\kappa(X,Y)$. Suppose that*

(C1.1) *for each $x \in D$, Gx is compactly closed in Y; and*

(C1.2) *for each $N \in \langle D\rangle$, $T(\Gamma_N) \subseteq G(N)$.*

Then $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

PROOF: Since X is compact and $T \in U_c^\kappa(X,Y)$, there exists $T' \in U_c(X,Y)$ such that $T'x \subseteq Tx$ for all $x \in X$. Since T' is upper semicontinuous with compact-values on X, it follows from Lemma 3 that $T'(X)$ is compact. Hence $T' \in U_c(X,Y) \subset KKM(X,Y)$ is compact. By (C1.2), for each $N \in \langle D\rangle$, $T'(\Gamma_N) \subseteq G(N)$. Then all the conditions for Theorem 1 are satisfied and it follows from Theorem 1 that $\overline{T'(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$. Therefore $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$. □

THEOREM 2. *Let (X, D) be a convex space, Y a Hausdorff space and $G : D \to 2^Y$, $T \in U_c^\kappa(X,Y)$ be set-valued maps satisfying the following*

(2.1) *for each $N \in \langle D\rangle$, $T(\mathrm{Co}\, N) \subseteq G(N)$; and*

(2.2) *for each $N \in \langle D\rangle$, and each $x \in N$, $Gx \cap T(\mathrm{Co}\, N)$ is relatively closed in $T(\mathrm{Co}\, N)$.*

Then, for each $N \in \langle D\rangle$, $T(\mathrm{Co}\, N) \cap \bigcap\{Gx : x \in N\} \neq \emptyset$.

PROOF: Let $\widetilde{N} \in \langle D\rangle$, and $Z = \mathrm{Co}\, \widetilde{N}$. Since $T \in U_c^\kappa(X,Y)$ and Z is compact, there exists $F \in U_c(Z,Y)$ such that $Fx \subseteq Tx$ for each $x \in Z$. As F is upper semicontinuous with compact values, it follows from Lemma 4 that $F(Z)$ is compact and F is compact. Let $G_1 : \widetilde{N} \to 2^Y$ be given by $G_1x = Gx \cap F(Z)$ for $x \in \widetilde{N}$. Then for each $N \in \langle\widetilde{N}\rangle$,

$F(\mathrm{Co}\,N) = F(\mathrm{Co}\,N) \cap F(Z) \subseteq T(\mathrm{Co}\,N) \cap F(Z) \subseteq G(N) \cap F(Z) = G_1(N)$. By (2.2), for each $x \in \widetilde{N}$, $Gx \cap T(Z) = Ax \cap T(Z)$, where $A : \widetilde{N} \to 2^Y$, Ax is closed for each $x \in \widetilde{N}$. Hence for each $x \in \widetilde{N}$, $G_1x = Gx \cap F(Z) = G(x) \cap T(Z) \cap F(Z) = Ax \cap T(Z) \cap F(Z) = Ax \cap F(Z)$ is closed in Y. This shows that for each $x \in \widetilde{N}$, G_1x is compactly closed in Y. We see $F \in U_c\big(Z, F(Z)\big) \subseteq KKM\big(Z, F(Z)\big)$. Replacing (D, X, Y, T, G) by $\big(\widetilde{N}, Z, F(Z), F, G_1\big)$ in Theorem 1, shows that $\overline{F(Z)} \cap \bigcap\{G_1x : x \in \widetilde{N}\} \neq \emptyset$. This implies $\overline{T(Z)} \cap \bigcap\{Gx : x \in \widetilde{N}\} \neq \emptyset$. Since $\widetilde{N} \in \langle D\rangle$ is arbitary, this completes the proof. □

COROLLARY 2. *Let X be a nonempty subset of a vector space, and $G : X \to 2^Y$, $T : \mathrm{Co}\,X \to ka(Y)$ set-valued maps satisfying the following*

(C2.1) *for each $N \in \langle X\rangle$, $T(\mathrm{Co}\,N) \subseteq G(N)$;*

(C2.2) *for each $N \in X$, $T|_{\mathrm{Co}\,N}$ is upper semicontinuous, where $\mathrm{Co}\,N$ is endowed with the Euclidean simplex topology; and*

(C2.3) *for each $N \in \langle X\rangle$, and each $x \in N$, $Gx \cap T(\mathrm{Co}\,N)$ is relatively closed in $T(\mathrm{Co}\,N)$.*

Then for each $N \in \langle X\rangle$, $T(\mathrm{Co}\,N) \cap \bigcap\{Gx : x \in N\} \neq \emptyset$.

PROOF: Let $\widetilde{X} \in \langle X\rangle$. By (C2.2), $\big(\mathrm{Co}\,\widetilde{N}, \widetilde{N}\big)$ is a convex space and $T|_{\mathrm{Co}\,\widetilde{N}} \in V\big(\mathrm{Co}\,\widetilde{N}, Y\big) \subseteq U_c^\kappa\big(\mathrm{Co}\,\widetilde{N}, Y\big)$. Then all conditions of Theorem 2 are satisfied and Corollary 2 follows immediately from Theorem 2. □

Applying Theorem 1, we generalise Fan [**5**, Theorem 6] and we improve [**3**, Theorem 8].

THEOREM 3. *Let X be a convex space, Y a Hausdorff space and $S : X \to 2^Y$, $T \in KKM(X, Y)$ maps satisfying the following conditions:*

(3.1) *for each compact subset C of X, $\overline{T(C)}$ is a compact subset of Y;*

(3.2) *for each $x \in X$, Sx is compactly closed in Y;*

(3.3) *for each $N \in \langle X\rangle$, $T(\mathrm{Co}\,N) \subseteq S(N)$; and*

(3.4) *there exists a compact convex subset X_0 of X and*

$$\bigcap\{Sx : x \in X_0\} \subseteq K.$$

Then $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$.

PROOF: Suppose that $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} = \emptyset$. Since K is compact, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $K \subseteq \big(\overline{T(X)}\big)^c \cup \Big(\bigcup\limits_{i=1}^n S^c x_i\Big)$, where $S^c x = Y \backslash Sx$. By (3.4), $K^c \subseteq \bigcup\limits_{x \in X_0} S^c x_i \subseteq \Big(\bigcup\limits_{x \in X_0} S^c x\Big) \cup \big(\overline{T(X)}\big)^c$. If we let $X_1 = \mathrm{Co}\big(X_0 \cup \{x_1, x_2, \ldots, x_n\}\big)$, then X_1 is a compact convex subset of X and $Y = \Big(\bigcup\limits_{x \in X_1} S^c x\Big) \cup \big(\overline{T(X)}\big)^c$, that is, $\overline{T(X)} \cap \bigcap\limits_{x \in X_1} Sx = \emptyset$. We define $F : X_1 \to 2^Y$ by $Fx = Sx \cap \overline{T(X_1)}$, $x \in X_1$.

Then (a) for each $x \in X_1$, Fx is a closed subset of $\overline{T(X_1)}$, (b) for each $N \in \langle X_1 \rangle$, $T(\operatorname{Co} N) \subseteq F(N)$. Since $T \in KKM(X,Y)$, it follows from Lemma 6 and (3.1) that $T|_{X_1} \in KKM(X_1,Y)$ is compact. By Theorem 1, we have $\overline{T|_{X_1}(X_1)} \cap \bigcap\{Sx : x \in X_1\} \neq \emptyset$. But $T|_{X_1}(X_1) \subseteq T(X)$, so we have $\overline{T(X)} \cap \bigcap\{Sx : x \in X_1\} \neq \emptyset$. This contradicts that $\overline{T(X)} \cap \bigcap\{Sx : x \in X_1\} = \emptyset$. Therefore $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$. ∎

REMARK 2. Theorem 3 improves [**3**, Theorem 8]. We prove Theorem 3 by applying Theorem 1, while [**3**, Theorem 8] is proved by applying the *KKM* property. From [**3**, Theorem 8] we only obtain the conclusion $\bigcap_{x \in X} Sx \neq \emptyset$.

COROLLARY 3. [5] *In a topological vector space, let Y be a convex set and $\emptyset \neq X \subset Y$. For each $x \in X$, let $F(x)$ be a relatively closed subset of Y such that the convex hull of every finite subset $\{x_1, x_2, \dots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^{n} F(x_i)$. If there is a nonempty subset X_0 of X such that the interection $\bigcap_{x \in X_0} F(x)$ is compact, and X_0 is contained in a compact convex subset of Y, then $\bigcap_{x \in X} F(x) \neq \emptyset$.*

PROOF: Take $T(x) = \{x\}$ and $K = \bigcap_{x \in X_0} F(x)$; then Corollary 3 follows immediately. ∎

COROLLARY 4. *Let X be a convex space, Y a Hausdorf space, and $S : X \to 2^Y$, $T \in KKM(X,Y)$ maps satisfying the following*

- (C4.1) *for each compact subset C of X, $\overline{T(C)}$ is compact;*
- (C4.2) *for each $x \in X$, Sx is compactly closed in Y;*
- (C4.3) *for each $N \in \langle X \rangle$, $T(\operatorname{Co} N) \subseteq S(N)$; and*
- (C4.4) *there is a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset X_1 of X and $\bigcap_{x \in X_0} Sx$ is a nonempty compact subset of Y.*

Then $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$.

PROOF: If we take $K = \bigcap_{x \in X_0} Sx$ in Theorem 3, then Corollary 4 follows immediately. ∎

THEOREM 4. *Let X be a convex space, Y a Hausdorff space, $S : X \to 2^Y$, $T \in U_c^{\kappa}(X,Y)$ satisfying*

- (4.1) *for each $x \in X$, Sx is compactly closed in Y;*
- (4.2) *for each $N \in \langle X \rangle$, $T(\operatorname{Co} N) \subseteq S(N)$; and*
- (4.3) *there exists a nonempty subset K of Y and a nonempty subset X_0 of X such that X_0 is contained in a compact convex subset X_1 of X and $\bigcap\{Sx : x \in X_0\} \subseteq K$.*

Then $K \cap \overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$.

PROOF: Let $N = \{x_1, x_2, \ldots, x_n\}$ be any finite subset of X, then it follows from (4.3) that $X_2 = \mathrm{Co}\,(X_1 \cup N)$ is a compact convex subset of X. By the assumption $T \in U_c^\kappa(X,Y)$, there exists $T' \in U_c(X_2, Y)$ such that $T'x \subseteq Tx$ for all $x \in X_2$ and $T'(X_2)$ is a compact subset of Y. Thus $T' \in U_c(X_2,Y) \subseteq KKM(X_2,Y)$ is compact. Then all the conditions of Theorem 1 are satisfied. It follows from Theorem 1 that $\overline{T'(X_2)} \cap \bigcap\{Sx : x \in X_2\} \neq \emptyset$. Hence $\overline{T'(X_2)} \cap \bigcap\Big\{Sx \cap \bigcap\{Sx : x \in X_1\} : x \in N\Big\} \neq \emptyset$. But $X_0 \subset X_1$, hence $\bigcap\{Sx : x \in X_1\} \subseteq \bigcap\{Sx : x \in X_0\} \subseteq K$. This shows that $\bigcap\Big\{Sx \cap \overline{T'(X_2)} \cap K : x \in N\Big\} \neq \emptyset$. Since for each $x \in X$, Sx is compactly closed in Y and $\overline{T'(X_2)}$ is compact, it follows that $\Big\{Sx \cap \overline{T'(X_2)} \bigcap K : x \in X\Big\}$ is a family of closed sets with the finite intersection property in the compact set $\overline{T'(X_2)} \cap K$. Therefore $\bigcap\Big\{Sx \cap \overline{T'(X_2)} \cap K : x \in X\Big\} \neq \emptyset$. Since $\overline{T'(X_2)} \subseteq T(X_2) \subseteq T(X)$, it follows that $K \cap \overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$. □

The following theorem generalises a fixed point theorem of Tarafdar [**15**].

THEOREM 5. *Let X be a convex space, Y a Hausdorff topological space, $T \in KKM(X,Y)$, $F : Y \to 2^X$ be set-valued maps satisfying*

(5.1) *for each compact set C of X, $\overline{T(C)}$ is compact;*

(5.2) *for each $y \in T(X)$, Fy is a nonempty convex subset of X;*

(5.3) *for each $x \in X$, $F^-(x)$ contains a compactly open subset O_x of Y;*

(5.4) $\bigcup\limits_{x \in X} O_x = Y$; *and*

(5.5) *there is a nonempty subset $X_0 \subset X$ such that X_0 is contained in a compact convex subset X_1 of X and the set $M = \bigcap\limits_{x \in X_0} O_x^c$ is compact (M may be empty) and O_x^c denotes the complement of O_x in Y.*

Then there exist $\overline{x} \in X$, and $\overline{y} \in T(\overline{x})$ such that $\overline{x} \in F(\overline{y})$.

PROOF: For each $x \in X$, we let $Sx = O_x^c$, then $S : X \to 2^Y$ and for each $x \in X$. Sx is compactly closed in Y. There are two cases:

CASE (1) $M = \emptyset$. In this case, if we take $X = X_0$ in Theorem 1, we have a finite subset $A = \{x_1, x_2, \ldots, x_n\}$ of X_0 such that $T(\mathrm{Co}\,A) \not\subseteq \bigcup\limits_{i=1}^n Sx_i$. This means that there exist $x_0 = \sum\limits_{i=1}^n \lambda_i x_i$, $\lambda_i \geqslant 0$, $i = 1, 2, \ldots, n$, $\sum\limits_{i=1}^n \lambda_i = 1$ and $y_0 \in Tx_0$ such that $y_0 \notin \bigcup\limits_{i=1}^n Sx_i = \bigcup\limits_{i=1}^n O_{x_i}^c$. Thus $y_0 \in O_{x_i} \subseteq F^-(x_i)$ for all $i = 1, 2, \ldots, n$. Hence $x_i \in F(y_0)$ for all $= 1, 2, \ldots, n$. But by (3.1), Fy_0 is convex, so we have $x_0 = \sum\limits_{i=1}^n \lambda_i x_i \in Fy_0$ and Theorem 5 is proved for the case $M = \bigcap\limits_{x \in X_0} O_x^c = \emptyset$.

CASE (2) $M \neq \emptyset$. We want to show that there exists a finite subset $A = \{x_1, x_2, \ldots, x_n\}$ of X such that $T(\mathrm{Co}\,A) \not\subseteq \bigcup\limits_{i=1}^n Sx_i$. Suppose that for each finite subset $B = \{u_1, u_2, \ldots, u_m\}$ of X, $T(\mathrm{Co}\,B) \subseteq \bigcup\limits_{i=1}^m Su_i$. Then it follows from Corollary 4 that $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq$

$\emptyset$. Hence $\bigcap_{x\in X} O_x^c = \bigcap_{x\in X} Sx \neq \emptyset$, therefore $\bigcup_{x\in X} O_x \neq Y$, which contradicts to the assumption (5.4) of this theorem. This shows that there exists a finite subset $A = \{x_1, x_2, \ldots, x_n\}$ of X such that $T(\mathrm{Co}\, A) \not\subseteq \bigcup_{i=1}^{n} Sx_i$. As in case (1), there exist $x_0 = \sum_{i=1}^{n} \lambda_i x_i$, $\lambda_1 \geqslant 0$, $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \lambda_i = 1$ and $y_0 \in Tx_0$ such that $y_0 \notin \bigcup_{i=1}^{n} Sx_i$. From this relation, we get that $x_0 \in Fy_0$ and $y_0 \in Tx_0$.

∎

Theorem 5 also gives a sufficient conditions for the existence of fixed points for the composition of two set-valued maps.

Corollary 5. *Under the assumption of Theorem 5, there exists $x_0 \in X$ such that $x_0 \in FTx_0$.*

Proof: It follows from Theorem 5, that there exist $x_0 \in X$, $y_0 \in Tx_0$ such that $x_0 \in Fy_0$. Hence $x_0 \in FTx_0$. ∎

Corollary 6. *Let X be a nonempty compact convex subset of a topological vector space, $T \in KKM(X, X)$ and $F : X \to 2^X$ be set-valued maps satisfying*

(C6.1) *for each $y \in X$, $F^-(y)$ contains a relatively open subset O_y of X (O_y could be empty);*

(C6.2) *for each $x \in X$, Fx is a nonempty subset of X; and*

(C6.3) $\bigcup_{y\in X} O_y = X$.

Then there exists point $x_0 \in X$, $y_0 \in Tx_0$ such that $x_0 \in Fy_0$.

Proof: Since X is compact and $\bigcup_{y\in X} O_y = X$, it follows that condition (5.5) holds automatically and Corollary 6 follows immediately from Theorem 5. ∎

Corollary 7. [15] *Let X be a nonempty compact convex subset of a topological vector space. Let $F : X \to 2^X$ be set-valued maps such that*

(C7.1) *for each $x \in X$, Fx is a nonempty convex subset of X;*

(C7.2) *for each $y \in X$, $F^-(y)$ contains a relatively open subset O_y of X (O_y may be empty for some y);*

(C7.3) $\bigcup_{y\in X} O_x = X$; *and*

(C7.4) *there exists a nonempty subset $X_0 \subset X$ such that X_0 is contained in a compact convex subset X_1 of X and $M = \bigcap_{x\in X_0} O_x^c$ is compact (M may be empty).*

Then there exists a point $x_0 \in X$ such that $x_0 \in Fx_0$.

Proof: If we define $T : X \to 2^X$ by $Tx = \{x\}$ and take $X = Y$ in Theorem 5, we prove Corollary 7. ∎

COROLLARY 8. [2] *Let X be a nonempty compact convex subset of a topological vector space. Let $F : X \to 2^X$ be set-valued maps such that*

(C8.1) *for each $y \in X$, $F^-(y)$ is open; and*

(C8.2) *for each $x \in X$, Fx is a nonempty convex subset of X.*

Then there is $x_0 \in X$ such that $x_0 \in Fx_0$.

PROOF: Since for each $x \in X$, Fx is a nonempty subset of X, there exists $y \in X$ such that $y \in Fx$. Hence $x \in F^-y$. This shows that $X = \bigcup_{y \in X} F^-y$. If we define $T : X \to 2^Y$ by $Tx = \{x\}$ for $x \in X$, then all the conditions of Corollary 7 are satisfied and Corollary 8 follows immediately from Corollary 7. □

REMARK 3. Corollary 4 can be proved by using Theorem 5. Suppose that all the conditions of Corollary 4 are satisfied; we want to show that $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$. Suppose on the contrary that $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} = \emptyset$. We define $H : \overline{T(X)} \to 2^X$ by $Hy = \{x \in X : y \notin Sx\}$. For each $x \in X$, we let $S^c x = Y \backslash Sx$ and $O_x = S^c x$. Clearly for each $y \in \overline{T(X)}$, $y \in \bigcup_{x \in X} S^c x$, hence $y \notin Sx_0$ for some $x_0 \in X$ and $H(y)$ is a nonempty subset of X. For each $x \in X$, $H^-(x) = \left\{y \in \overline{T(X)} : y \notin Sx\right\} = S^c x \cap \overline{T(X)} = O_x \cap \overline{T(X)}$ is compactly open in $\overline{T(X)}$. Now we denote $\hat{O}_x = O_x \cap \overline{T(X)}$. Let $F : \overline{T(X)} \to 2^X$ be defined by $Fy = \operatorname{Co}[Hy]$ for each $y \in \overline{T(X)}$. Then for each $y \in \overline{T(X)}$, Fy is a nonempty convex subset of X and for each $x \in X$, $F^-(x) \supseteq H^-(x) = \hat{O}_x$. Since $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} = \emptyset$, it follows that $\overline{T(X)} \subset \bigcup_{x \in X} S^c x$ and $\overline{T(X)} = \bigcup_{x \in X} \left[S^c x \cap \overline{T(X)}\right] = \bigcup_{x \in X} \left[O_x \cap \overline{T(X)}\right] = \bigcup_{x \in X} \hat{O}_x$. We denote by $\hat{O}_x^c$ the complement of $\hat{O}_x$ in $\overline{T(X)}$. By (C4.2) and (C4.4), $\bigcap_{x \in X_0} \hat{O}_x^c = \bigcap_{x \in X_0} \left[\overline{T(X)} \backslash \hat{O}_x\right] = \overline{T(X)} \cap \bigcap_{x \in X_0} O_x^c = \overline{T(X)} \cap \bigcap_{x \in X_0} Sx$ is compact in $\overline{T(X)}$. Then it follows from Theorem 5 that there exists $\overline{x} \in X$, $\overline{y} \in T(\overline{x})$ such that $\overline{x} \in F\overline{y} = \operatorname{Co}[H\overline{y}]$. This implies there exists $A = \{x_1, x_2, \dots, x_n\} \subseteq H(\overline{y})$, $\lambda_i \geqslant 0$, $i = 1, 2 \dots, n$, $\sum_{i=1}^{n} \lambda_i = 1$ such that $\overline{x} = \sum_{i=1}^{n} \lambda_i x_i$. Since $x_i \in H(\overline{y})$ for all $i = 1, 2, \dots, n$, it follows that $\overline{y} \notin Sx_i$ for all $i = 1, 2, \dots, n$. Therefore $T(\operatorname{Co} A) \not\subseteq \bigcup_{i=1}^{n} Sx_i$. This contradicts the assumption (C4.3) of Corollary 4. Hence $\overline{T(X)} \cap \bigcap\{Sx : x \in X\} \neq \emptyset$ and Corollary 4 is proved.

THEOREM 6. *Let X be a convex space, Y a Hausdorff topological space, $T \in U_c^\kappa(X, Y)$, $F : Y \to 2^X$ be set-valued maps satisfying*

(6.1) *for each $y \in T(X)$, Fy is a nonempty convex subset of X;*

(6.2) *for each $x \in X$, $F^-(x)$ contains an compactly open subset O_x of Y;*

(6.3) $\bigcup_{y \in X} O_x = Y$; *and*

(6.4) *there exists a nonempty subset $X_0 \subseteq X$ such that X_0 is contained in a compact convex subset X_1 of X and the set $M = \bigcap_{x \in X_0} O_x^c$ is compact (M may be empty) and O_x^c denotes the complement of O_x in Y.*

Then there exist $\overline{x} \in X$ and $\overline{y} \in T\overline{x}$ such that $\overline{x} \in F\overline{y}$.

PROOF: For each $x \in X$, we let $Sx = O_x^c$. Then $S : X \to 2^Y$ and for each $x \in X$. Sx is compactly closed in Y. There are two cases.

CASE (1) $M = \emptyset$. In this case, we use Corollary 1 and follow the same argument as in Theorem 5.

CASE (2) $M \neq \emptyset$. In this case, we use Theorem 4 and follow the same argument as in Theorem 5.

□

REMARK 4. In Theorem 5, we assume that $T \in KKM(X,Y)$ and $\overline{T(C)}$ is compact for each compact set C of X, but in Theorem 6, we assume only that $T \in U_c^\kappa(X,Y)$.

THEOREM 7. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $T : X \to 2^Y$ be compact and closed and $G : D \multimap 2^Y$. Suppose that*

(7.1) *for each $x \in D$, Gx is compactly closed;*

(7.2) *for any $N \in \langle D \rangle$, $T(\Gamma_N) \subseteq G(N)$; and*

(7.3) *there exist a nonempty compact subset K of Y and for each $N \in \langle D \rangle$, a compact, G-convex subset L_N of X containing N such that $T(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$, and $T \in G$-KKM (L_N, Y).*

Then $\overline{T(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

PROOF: Suppose that $\overline{T(X)} \cap K \cap \bigcap\{Gx : x \in D\} = \emptyset$. Let $Sx = Y \backslash Gx$, then $\overline{T(X)} \cap K \subset S(D)$. Since $\overline{T(X)} \cap K$ is compact and for each $x \in D$, Sx is compactly open, it follows that there exists $N \in \langle D \rangle$ such that $\overline{T(X)} \cap K \subseteq S(N)$. By (7.3), there exists a compact G-convex subset L_N of X containing N such that $T(L_N) \backslash K \subseteq S(L_N \cap D)$. Hence $T(L_N) \subseteq S(L_N \cap D)$. Since T is compact and closed, it follows from Lemma 2 that T is upper semicontinuous We want to show that for each $x \in X$, Tx is compact. Let $y \in \overline{T(x)}$, then there exists a net $\{y_\alpha\}$ in Tx such that $y_\alpha \to y$. Since T is closed, it follows that $y \in Tx$ and Tx is closed. By assumption T is compact, hence $\overline{T(X)}$ is a compact set. But $Tx \subseteq \overline{T(X)}$ and Tx is closed for each $x \in X$. This shows that Tx is compact for each $x \in X$. Since T is upper semicontinuous with compact values and L_N is compact, it follows from Lemma 3 that $T(L_N)$ is compact. Therefore $\overline{T(L_N)} = T(L_N) \subseteq S(L_N \cap D)$. Thus $\overline{T(L_N)} \cap \bigcap\{Gx : x \in L_N \cap D\} = \emptyset$. It follows from Theorem 1 with $(T|_{L_N}, G|_{L_N \cap D}, L_N, L_N \cap D)$ replacing (T, G, X, D), that there exists $M \in \langle L_N \cap D \rangle \subseteq \langle D \rangle$ such that $T(\Gamma_M) \not\subseteq G(M)$. This contradicts (7.2). Therefore $\overline{T(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$. □

COROLLARY 9. [12] *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, and $T \in U_c^\kappa(X,Y)$. Let $G : D \to 2^Y$ be a map such that*

(C9.1) *for each $x \in D$, Gx is compactly closed in Y;*

(C9.2) *for any $N \in \langle D \rangle$, $T(\Gamma_N) \subseteq G(N)$; and*

(C9.3) *there exist a nonempty compact subset K of Y and for each $N \in \langle D \rangle$, a compact G-convex subset L_N of X containing N such that $T(L_N) \cap \bigcap\{Gx : x \in L_N \cap D\} \subset K$.*

Then $\overline{T(X)} \cap K \cap \bigcap\{Gx : x \in D\} \neq \emptyset$.

PROOF: Since $T \in U_c^\kappa(X,Y) \subset G\text{-}KKM(X,Y)$, it follows from Lemma 6 that $T|_{L_N} \in G\text{-}KKM(L_N, Y)$ and the conclusion of Corollary 9 follows from Theorem 7. ∎

4. GENERALISED G-KKM THEOREMS

As a consequence of the generalised *G-KKM* theorem, we prove a generalisation of the Ky Fan matching theorem.

THEOREM 8. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space, $S : D \to 2^Y$ and $T \in G\text{-}KKM(X,Y)$ be compact. Suppose that*

(8.1) *for each $x \in D$, Sx is compactly open in Y; and*

(8.2) $\overline{T(X)} \subset S(D)$.

Then there exists $M \in \langle D \rangle$ such that $T(\Gamma_M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$.

PROOF: Suppose that the conclusion of Theorem 8 is false. Then for any $N \in \langle D \rangle$, $T(\Gamma_N) \cap \bigcap\{Sx : x \in N\} = \emptyset$. Therefore $T(\Gamma_N) \subseteq \bigcup\{Gs : s \in N\} = G(N)$, where $Gx = Y \backslash Sx$. By (8.1), for each $x \in D$, Gx is compactly closed in Y. Then all the conditions of Theorem 1 are satisfied. It follows from Theorem 1 that $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} \neq \emptyset$. Hence $\overline{T(X)} \not\subseteq S(D)$, but this contradicts (8.2). Thus there exists $M \in \langle D \rangle$ such that $T(\Gamma_M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$. ∎

COROLLARY 10. [8] *Let D be a nonempty subset in a compact convex space X, Y a topological space, and $A : D \to 2^Y$ a set-valued map satisfying*

(C10.1) *for each $x \in D$, Ax is compactly open in Y; and*

(C10.2) $A(D) = Y$.

Then for any $x \in C(X,Y)$, there exist a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and $x_0 \in \operatorname{Co}\{x_1, \ldots, x_n\}$ such that $sx_0 \in \bigcap_{i=1}^{n} Ax_i$.

PROOF: Since X is compact and $s \in C(X,Y)$, it follows that $s(X)$ is compact. Hence $s \in C(X,Y) \subseteq KKM(X,Y)$ is compact. By (C10.2), $\overline{s(X)} = s(X) \subseteq Y \subseteq A(D)$. It follows from Theorem 8, that there exist a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and $x_0 \in \operatorname{Co}\{x_1, \ldots, x_n\}$ such that $sx_0 \in \bigcap_{i=1}^{n} Ax_i$. ∎

COROLLARY 11. [5] *In a topological vector space, let Y be a convex set and let X be a nonempty subset of Y. For each $x \in X$, let Ax be relative open in Y such that $\bigcup_{x \in X} Ax = Y$. If X is contained in a compact convex subset C of Y, then there exist a nonempty, finite subset $\{x_1, x_2, \ldots, x_n\}$ of X and $x_0 \in \{x_1, \ldots, x_n\}$ such that $x_0 \in \bigcap_{i=1}^{n} Ax_i$.*

PROOF: Let $Tx = \{x\}$, then $T(C) = C$ is compact, and T is compact. $\overline{T(C)} = C \subseteq Y \subseteq A(X)$. Then it follows from Theorem 8 that there exist a finite subset $\{x_1, x_2, \dots, x_n\}$ of X and $x_0 \in \{x_1, \dots, x_n\}$ such that $x_0 \in \bigcap_{i=1}^{n} Ax_i$. □

REMARK 5. Theorems 1 and 8 are equivalent.

We saw that Theorem 8 can be proved by using Theorem 1. Now we prove Theorem 1 from Theorem 8. Suppose that $\overline{T(X)} \cap \bigcap\{Gx : x \in D\} = \emptyset$. Let $Sx = Y \backslash Gx$. Then Sx is compactly open and $\overline{T(X)} \subset S(D)$. It follows from Theorem 8, that there exists $M \in \langle D \rangle$ such that $T(\Gamma_M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$. Hence $T(\Gamma_M) \not\subseteq G(M)$. This contradicts (1.2). Thus the conclusion of Theorem 1 holds.

THEOREM 9. *Let $(X, D; \Gamma)$ be a G-convex space, Y a Hausdorff space and $T : X \to 2^Y$ be compact and closed. Suppose that*

(9.1) *for each $x \in D$, Sx is compactly open;*

(9.2) *there exists a nonempty compact subset K of Y such that $\overline{T(X)} \subset S(D)$; and*

(9.3) *for each $N \in \langle D \rangle$, there exists a compact G-convex subset L_N of X containing N such that $T(L_N) \backslash K \subseteq S(L_N \cap D)$, and $T \in G\text{-}KKM(L_N, Y)$.*

Then there exists $M \in \langle D \rangle$ such that $T(\Gamma_M) \cap \bigcap\{Sx : x \in M\} \neq \emptyset$.

PROOF: Suppose that for any $N \in \langle D \rangle$. $T(\Gamma_N) \cap \bigcap\{Sx : x \in N\} = \emptyset$. Let $Gx = Y \backslash Sx$. Then by applying Corollary 9 and following an argument as in Theorem 8, we prove Theorem 9. □

REFERENCES

[1] J.P. Aubin, A. Cellina, *Differential inclusions* (Springer-Verlag, Berlin, Heidlberg, New York, 1984).

[2] F.E. Browder, 'On nonlinear monotone operators and convex set in Banach space', *Bull. Amer. Math. Soc.* **71** (1965), 780–785.

[3] T.H. Chang and C.L. Yen, 'KKM property and fixed point theorems', *J. Math. Anal. Appl.* **203** (1996), 224–235.

[4] K. Fan, 'A generalization of Tychonoff's fixed point theorem', *Math. Ann.* **142** (1961), 306–310.

[5] K. Fan, 'Some properties of convex sets related to fixed point theorem', *Math. Ann.* **266** (1985), 519–537.

[6] B. Knaster, K. Kuratowski and S. Mazurkiewicz, 'Ein Beweis Fixpunktsatzes fur n-dimensionale Simplexe', *Fund. Math.* **1** (1929), 132–137.

[7] M. Lassonde, 'On the use of KKM multifunctions in fixed point theory and related topics', *J. Math. Anal. Appl.* **97** (1983), 151–201.

[8] S. Park, 'Generalizations of Ky Fan's matching theorems and their applications', *J. Math. Anal. Appl.* **141** (1989), 164–176.

[9] S. Park, 'A unfied approach to generalizations of the KKM-type theorems related to acyclic maps', *Numer. Funct. Anal. Optim.* **15** (1994), 105–119.

[10] S. Park, 'Foundations of the KKM theory via coincidence of composites of upper semicontinuous maps', *J. Korean Math. Soc.* **31** (1994), 493–519.

[11] S. Park, 'On minimax inequalities on spaces having certain contractible subsets', *Bull. Austral. Math. Soc.* **47** (1993), 25–40.

[12] S. Park and H. Kim, 'Coincidence theorems for admissible multifunctions on generalized convex spaces', *J. Math. Anal. Appl.* **197** (1996), 173–187.

[13] S. Park and H. Kim, 'Fundations of the KKM theory on generalized convex spaces', *J. Math. Anal. Appl.* **209** (1997), 551–571.

[14] N. Shioji, 'A further generalization of the Knaster-Kuratowski-Mazurkiewicz theorem', *Proc. Amer. Math. Soc.* **111** (1991), 187–195.

[15] E. Tarafdar, 'A fixed point theorem equivalent to the Fan-Knaster-Kuratwski-Mazurkiewicz theorem', *J. Math. Anal. Appl.* **128** (1987), 475–479.

[16] E. Tarafdar, 'On nonlinear variational inequalities', *Proc. Amer. Math. Soc.* **67** (1977), 95–98.

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