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System of coincidence theorems with applications ☆

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Abstract

In this paper, we establish systems of coincidence theorems from which solution of system of inequalities and system of minimax theorems was established in this paper.

Keywords: Transfer (compactly transfer) open valued; Coincidence point; Compactly closed (open); Compact closure (interior); Upper (lower) semicontinuous

1. Introduction

In 1937 von Neumann [18] established the well-known coincidence theorem. Since then, there have been a lot of generalization and applications, see [3,8,12], [12, pp. 96–97] and references therein. Recently Deguire and Lassonde [5] and Deguire et al. [6] studied some system of coincidence theorems of KF families [6] and give some of its applications. In [5,6], the authors only established the existence theorems of a pair of multimaps in two families of multimaps. Recently, Ansari et al. [2], Yu and Lin [20], and Lin and Chen [14] studied the coincidence theorems for two families of multimaps. Ansari et al. [2] and Lin and Chen [14] also gave some applications to the study of the equilibrium problem. In this paper, we establish some systems of coincidence theorems, from which the existence theorem of system of inequalities and system of minimax theorems are established. The system of minimax theorems we establish in this paper are quite different from the minimax

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theorem in [5]. Our results on system of coincidence theorems are different from [2,5,6,14, 20] and our results include some results of [3] and many well-known results in the literature as special cases.

2. Preliminaries

Let X and Y be nonempty sets. A multimap $T : X \multimap Y$ is a function from X into the power set of Y. Let $A \subset X$, $x \in X$, and $y \in Y$. We define $T(A) = \bigcup\{T(x) \mid x \in A\}$; $x \in T^-(y)$ if and only if $y \in T(x)$.

For topological spaces X and Y, $A \subset X$, we denote $\text{int}_X A$ to be the interior of A in X; A is said to be compactly closed (respectively, open) if for every nonempty compact subset K of X, $A \cap K$ is closed (respectively, open) in K. The compact closure of A (see [7]) is defined by

$$\text{ccl}\, A = \bigcap\{B \subset X \mid A \subset B \text{ and } B \text{ is compactly closed in } X\}$$

and the compactly interior of A is defined by

$$\text{cint}\, A = \bigcup\{B \subset X \mid B \subset A \text{ and } B \text{ is compactly open in } X\}.$$

It is easy to see that

$$\text{ccl}(X \setminus A) = X \setminus \text{cint}\, A.$$

Let $T : X \multimap Y$, T is said to be transfer compactly closed valued (respectively, transfer closed valued) on X [7,17], if for every $x \in X$, $y \in T(x)$, there exists $x' \in X$ such that $y \notin \text{ccl}\, T(x')$ (respectively, $y \notin \text{cl}\, T(x')$); T is said to be transfer compactly open valued (respectively, transfer open valued) on X if for every $x \in X$, $y \in T(x)$, there exists $x' \in X$ such that $y \in \text{cint}\, T(x')$ (respectively, $y \in \text{int}\, T(x')$).

Definition [17]. Let X and Y be two topological spaces, $f : X \times Y \to \mathbb{R} \cup \{-\infty, \infty\}$ a function, f is said to be transfer compactly (respectively, transfer) l.s.c. in y if for each $y \in Y$ and each $\gamma \in \mathbb{R}$ with $y \in \{u \in Y\colon\ f(x,u) > \gamma\}$, there exists an $x' \in X$ such that $y \in \text{cint}\{y \in Y\colon\ f(x', y) > \gamma\}$ (respectively, $y \in \text{int}\{y \in Y\colon\ f(x', y) > \gamma\}$); f is said to be transfer compactly (respectively, transfer) u.s.c. in y if $-f$ is transfer compactly (respectively, transfer) l.s.c. in y.

Remark 1. (a) It is easy to see that if for each $x \in X$, $y \to f(x, y)$ is l.s.c., then f is transfer l.s.c. in y.

(b) Let $F : X \multimap Y$ be defined by $F(x) = \{y \in Y \mid f(x, y) > \gamma\}$. If $f(x, y)$ is transfer compactly (respectively, transfer) l.s.c. in y, then $F : X \multimap Y$ is transfer compactly (respectively, transfer) open valued on X.

Following the method of Chang et al. [4], we have the following lemma.

Lemma 2.1. *Let X and Y be two topological spaces and $G : X \multimap Y$ be a multivalued map. Then G is transfer compactly open valued if and only if*

$$\bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{cint} G(x).$$

Proof. Suppose that G is transfer compactly open valued. Let $y \in \bigcup_{x\in X} G(x)$, then there exists $x_1 \in X$ such that $y \in G(x_1)$. Since G is transfer compactly open valued, there exists $x' \in X$ such that $y \in \operatorname{cint} G(x') \subset \bigcup_{x\in X} \operatorname{cint} G(x)$. Therefore, $\bigcup_{x\in X} G(x) \subset \bigcup_{x\in X} \operatorname{cint} G(x)$. Since $\operatorname{cint} G(x) \subset G(x)$, $\bigcup_{x\in X} \operatorname{cint} G(x) \subset \bigcup_{x\in X} G(x)$. Therefore, $\bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{cint} G(x)$. Conversely, if $\bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{cint} G(x)$. For each $x \in X$ and $y \in G(x)$, we have $y \in \bigcup_{x\in X} G(x) = \bigcup_{x\in X} \operatorname{cint} G(x)$. Therefore, there exists $x' \in X$ such that $y \in \operatorname{cint} G(x')$. Hence G is transfer compactly open valued. □

Applying Lemma 2.1 and following the same argument of Proposition 1 [13], we have the following lemma.

Lemma 2.2. *Let X and Y be two topological spaces and $G : X \multimap Y$ be a multivalued map. Then the following statements are equivalent*:

(i) *$G(x)$ is nonempty for each $x \in X$ and $G^- : Y \multimap X$ is transfer compactly open valued*;
(ii) $X = \bigcup_{y\in Y} \operatorname{cint} G^-(y)$.

Proof. (ii) ⇒ (i). Suppose $X = \bigcup_{y\in Y} \operatorname{cint} G^-(y)$. Then for each $x \in X$, we have $x \in \bigcup_{y\in Y} \operatorname{cint} G^-(y)$. There exists $y \in Y$ such that $x \in \operatorname{cint} G^-(y) \subset G^-(y)$. Therefore $y \in G(x)$ and $G(x)$ is nonempty. Since $X = \bigcup_{y\in Y} \operatorname{cint} G^-(y) \subseteq \bigcup_{y\in Y} G^-(y) \subseteq X$, $X = \bigcup_{y\in Y} \operatorname{cint} G^-(y) = \bigcup_{y\in Y} G^-(y)$. By Lemma 2.1, $G^- : Y \multimap X$ is transfer compactly open valued and (i) is true. Conversely, suppose that (i) is true, then for each $x \in X$, $G(x)$ is nonempty and $G^- : Y \multimap X$ is transfer compactly open valued. Therefore $X = \bigcup_{y\in X} G^-(y)$ and by Lemma 2.1 we have that $\bigcup_{y\in X} G^-(y) = \bigcup_{y\in X} \operatorname{cint} G^-(y)$. From this, $X = \bigcup_{y\in X} \operatorname{cint} G^-(y)$ and (ii) is true. □

The following example shows a set which is transfer compactly open, but it is not compactly open.

Example [19]. Let $X = Y = [0, 2)$ and $F : X \multimap Y$ be defined by $F(x) = [x, 2)$. Then $F^-(y) = [0, y]$ and $X = \bigcup\{\operatorname{int}_X F^-(y)\colon y \in Y\} \subseteq \bigcup\{\operatorname{cint} F^-(y)\colon y \in Y\} \subset X$. Therefore $X = \bigcup\{\operatorname{cint} F^-(y)\colon y \in Y\}$. By Lemma 2, $F^- : Y \multimap X$ is transfer compactly open valued, but $F^-(y) = [0, y]$ is not compactly open.

Let X be a nonempty convex subset of a real topological vector space E. A function $f : X \to \mathbb{R}$ is said to be quasiconvex if for each $x, y \in X$, $\lambda \in [0, 1]$, $f[\lambda x + (1 - \lambda)y] \leqslant \max\{f(x), f(y)\}$; f is said to be quasiconcave if $-f$ is quasiconvex.

Throughout this paper, all topological spaces are assumed to be Hausdorff and topological vector spaces will be denoted by t.v.s.

3. System of coincidence theorem

Theorem 3.1. *Let I be an index set, $\{E_i\}_{i\in I}$ be a family of t.v.s. For each $i \in I$, let X_i be a nonempty convex subset of E_i, let $S_i, F_i : X^i = \prod_{j\in I_{j\neq i}} X_j \multimap X_i$ and $H_i, T_i : X_i \multimap X^i$ be multimaps satisfying the following conditions*:

(i) *For all $x^i \in X^i$, $\operatorname{co} S_i(x^i) \subset F_i(x^i)$;*
(ii) *S_i has nonempty values on each point of X^i and S_i^- is transfer compactly open valued on X_i;*
(iii) *If X^i is not compact, there exists a nonempty compact subset $K(i)$ of X^i such that for each finite subset P_i of X_i, there exists a compact convex subset L_{P_i} of X_i containing P_i such that $X^i\backslash K(i) \subset \bigcup\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in L_{P_i}\}$;*
(iv) *For each $x_i \in X_i$, $\operatorname{co} H_i(x_i) \subset T_i(x_i)$;*
(v) *H_i has nonempty values on X_i and H_i^- is transfer compactly open valued on X^i;*
(vi) *If X_i is not compact, there exists a nonempty compact subset $M(i)$ of X_i such that for each finite subset Q^i of X^i, there exists a compact convex subset L_{Q^i} of X^i such that*

$$X_i\backslash M(i) \subset \bigcup\left\{\operatorname{cint} H_i^-(x^i)\colon\ x^i \in L_{Q^i}\right\}.$$

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X = \prod_{i\in I} X_i$ such that for each $i \in I$, $\bar{y}_i \in F_i(\bar{x}^i)$ and $\bar{x}^i \in T_i(\bar{y}_i)$.

Proof. By (ii) and Lemma 2.2, $X^i = \bigcup\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in X_i\}$ for each $i \in I$. Since $K(i)$ is a compact subset of X^i, there exists a finite subset P_i of X_i such that

$$K(i) \subset \bigcup\left\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in P_i\right\}. \tag{1}$$

Similarly, by (v) there exists finite subset Q^i of X^i such that

$$M(i) \subset \bigcup\left\{\operatorname{cint} H_i^-(x^i)\colon\ x^i \in Q_i\right\}. \tag{2}$$

By (iii),

$$L_{Q^i}\backslash K(i) \subset X^i\backslash K(i) \subset \bigcup\left\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in L_{P_i}\right\}. \tag{3}$$

By (1) and (3),

$$L_{Q_i} \subset \bigcup\left\{\operatorname{cint} S_i^-(x_i)\colon\ x_i \in L_{P_i}\right\}. \tag{4}$$

Similarly by (2) and (iv),

$$L_{P_i} \subset \bigcup\left\{\operatorname{cint} H_i^-(x^i)\colon\ x^i \in L_{Q_i}\right\}. \tag{5}$$

By (5), there exists $\{a_{i1}, \ldots, a_{im_i}\}$ in L_{Q_i} such that

$$L_{P_i} \subset \bigcup_{j=1}^{m_i} \operatorname{cint} H_i^-(a_{ij}). \tag{6}$$

By (4), there exists $\{b_{i1}, \ldots, b_{i\ell_i}\}$ in L_{P_i} such that

$$L_{Q_i} \subset \bigcup_{j=1}^{\ell_i} \operatorname{cint} S_i^-(b_{ij}). \tag{7}$$

Let $A^i = \operatorname{co}\{a_{i1}, \ldots, a_{im_i}\}$, $B_i = \operatorname{co}\{b_{i1}, \ldots, b_{i\ell_i}\}$, and $B = \prod_{i\in I} B_i$. Let W_i be the vector subspace of E_i generated by $\{b_{i1}, \ldots, b_{is}\}$. Since W_i is finite dimensional, W_i and $W = \prod_{i\in I} W_i$ are locally convex t.v.s. A^i, B_i, and B are compact convex subsets of L_{Q_i}, W_i, and W, respectively.

By (7),

$$A^i = \bigcup_{j=1}^{\ell_i} \bigl(\operatorname{cint} S_i^-(b_{ij})\bigr) \cap A^i. \tag{8}$$

By (6),

$$B_i = \bigcup_{j=1}^{m_i} \bigl(\operatorname{cint} H_i^-(a_{ij})\bigr) \cap B_i. \tag{9}$$

By (8) and using partition of unity, there exist continuous functions $\lambda_{i1}, \ldots, \lambda_{i\ell_i} : A^i \to [0,1]$ such that $\sum_{k=1}^{m_i} \lambda_{ik}(x^i) = 1$ and for each $k = 1, \ldots, \ell_i$, $\lambda_{ik}(x^i) = 0$ for $x^i \notin \operatorname{cint} S_i^-(b_{ik}) \cap A^i$. For each $i \in I$, we define $f_i : A^i \to B_i$ by

$$f_i(x^i) = \sum_{k=1}^{\ell_i} \lambda_{ik}(x^i) b_{ik} \quad \text{for } x^i \in A^i.$$

For each $x^i \in X^i$ and each k with $\lambda_{ik}(x^i) \neq 0$, we have $x^i \in \operatorname{cint} S_i^-(b_{ik}) \cap A^i \subseteq S_i^-(b_{ik})$. Therefore, $b_{ik} \in S_i(x^i)$ for each $i \in I$. By (i), for each $i \in I$, $f_i(x^i) \in \operatorname{co} S_i(x^i) \subseteq F_i(x^i)$ for all $x^i \in A^i$ and $f_i : A^i \to B_i$ is a continuous function. Similarly by (9), for each $i \in I$, there exists a continuous function $g_i : B_i \to A^i$ such that $g_i(y_i) \in \operatorname{co} H_i(y_i) \subset T_i(y_i)$ for all $y_i \in B_i$. Let $h : B \to B$ be defined by $h(x) = \prod_{i\in I} f_i(g_i(x_i))$.

B is a compact convex subset of the locally convex t.v.s. $\prod_{i\in I} W_i = W$. Then by Tychnoff's fixed point theorem, there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in B \subseteq X$ such that $\bar{y} = h(\bar{y}) = \prod_{i\in I} f_i(g_i(\bar{y}_i))$.Therefore $\bar{y}_i = f_i(g_i(\bar{y}_i))$ for all $i \in I$. Let $\bar{x}^i = g_i(\bar{y}_i)$. Then $\bar{x}^i = g_i(\bar{y}_i) \in T_i(\bar{y}_i)$, $\bar{y}_i \in f_i(\bar{x}^i) \in F_i(\bar{x}^i)$ for all $i \in I$ and $\bar{x} = (\bar{x}^i)_{i\in I} \in X$. □

As a consequence of Lemma 2.2 and Theorem 3.1, we have the following theorem.

Theorem 3.2. *Let I be an index set, $\{E_i\}_{i\in I}$ be a family of t.v.s. For each $i \in I$, let X_i be a nonempty convex subset of E_i, $S_i, F_i : X^i = \prod_{j\in I_{j\neq i}} X_j \multimap X_i$ and $H_i, T_i : X_i \multimap X^i$ be multimaps satisfying the following conditions*:

(i) *For all $x^i \in X^i$*, $\operatorname{co}(S_i(x^i)) \subseteq F_i(x^i)$;
(ii) $X^i = \bigcup\{\operatorname{int}_{X^i} S_i^-(x_i)\colon\ x_i \in X_i\}$;

(iii) *If X^i is not compact, there exists a compact subset $K(i)$ of X^i such that for each finite subset P_i of X_i, there exists a compact convex subset L_{P_i} of X_i containing P_i such that $X^i \setminus K(i) \subset \bigcup\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in L_{P_i}\}$;*
(iv) *For each $x_i \in X_i$, $\operatorname{co} H_i(x_i) \subseteq T_i(x_i)$;*
(v) $X_i = \bigcup\{\operatorname{int}_{X_i} H_i^-(x^i)\colon\ x^i \in X^i\}$;
(vi) *If X_i is not compact, there exists a nonempty compact subset $M(i)$ of X_i such that for each finite subset Q^i of X^i, there exists a compact convex subset L_{Q^i} of X^i such that $X_i \setminus M(i) \subset \bigcup\{\operatorname{cint} H_i^-(x^i)\colon\ x^i \in L_{Q^i}\}$.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that for each $i \in I$, $\bar{y}_i \in F_i(\bar{x}^i)$ and $\bar{x}^i \in T_i(\bar{y}_i)$.

Proof. Since $\operatorname{int}_{X_i} S_i^-(x_i) \subset \operatorname{cint} S_i^-(x_i)$, by (ii) we have $X^i = \bigcup\{\operatorname{int}_{X^i} S_i^-(x_i)\colon\ x_i \in X_i\} \subset \bigcup\{\operatorname{cint} S_i^-(x_i)\colon\ x_i \in X_i\} \subset X^i$. Therefore, $X^i = \bigcup\{\operatorname{cint} S_i^-(x_i)\colon\ x_i \in X_i\}$. It follows from Lemma 2.2 that $S_i^- : X_i \multimap X^i$ is transfer compactly open on X_i and $S_i(x^i)$ is nonempty for all $x^i \in X^i$.

Similarly, by (v) for all $x_i \in X_i$, $H_i(x_i)$ is nonempty and $H_i^- : X^i \multimap X_i$ is transfer compactly open on X^i. Since $\operatorname{int}_{X^i} S_i^-(y_i) \subseteq \operatorname{cint} S_i^-(y_i)$, it follows from (iii) that $X^i \setminus K(i) \subset \bigcup\{\operatorname{cint} S_i^-(y_i)\colon\ y_i \in L_{P_i}\}$.

Similarly, $X_i \setminus M(i) \subset \bigcup\{\operatorname{cint} H_i^-(x^i)\colon\ x^i \in L_{Q^i}\}$. Then all the conditions of Theorem 3.1 are satisfied and the conclusion of Theorem 3.2 follows from Theorem 3.1. □

Remark 2. If for all $i \in I$, $x^i \in X^i$, $S_i(x^i)$ is nonempty, and $S_i^-(y_i)$ is open for all $y_i \in X_i$, then condition (ii) of Theorem 3.2 is satisfied. Similarly, for all $i \in I$ and $x_i \in X_i$, $H_i(x_i)$ is nonempty and $H_i^-(x^i)$ is open for all $x^i \in X^i$, then condition (v) of Theorem 3.2 is satisfied.

If $I = \{1, 2\}$ and for all $i \in I$, X_i is compact, then by Theorem 3.2, it is easy to show the Fan's coincidence theorem.

Corollary 3.1 [9]. *Let $X \subset E$ and $Y \subset Z$ be nonempty compact convex sets in the t.v.s. E and Z, respectively. Let $A, B : X \multimap Y$ be two multivalued maps such that*

(i) *Ax is open and $B(x)$ is a nonempty convex set for each $x \in X$;*
(ii) *$B^- y$ is open and $A^- y$ is a nonempty convex set for each $y \in Y$.*

Then there exists $x_0 \in X$ such that $Ax_0 \cap Bx_0 = \emptyset$.

Remark 3. (a) The coercivity conditions used in Theorem 10 in [6] and Theorem 3.1 are different. Theorem 10 in [6] assume one multivalued map in each family satisfies the coercivity conditions. The conclusions of Theorem 10 in [6] and Theorem 4a in [5] are that there exist one pair of multivalued maps among two families of multivalued maps having a coincidence point. The proofs of Theorem 10 in [5] and Theorem 3.1 are different.

(b) In [2,14,20], the authors also establish system of coincidence theorems for two families of multivalued maps, but the conditions, proofs and the conclusions of Theorems 3.1–3.6 in [14], Theorem 8 in [20], and Theorem 2.1 in [2] are different from Theorems 3.1 and 3.2.

(c) It is very easy to see the coercivity condition in Theorem 3.1 or 3.2 is weaker than the coercivity conditions in [1,10,11,15].

4. Applications of system of coincidence theorems

Theorem 4.1. *Let I be an index sets, $\{E_i\}_{i\in I}$ be family of t.v.s. For each $i \in I$, let X_i be a nonempty convex subsets of E_i, Z_i be a real t.v.s., C_i a closed convex solid cone (i.e., $\operatorname{int} C_i \neq \emptyset$) in Z_i, and $A_i, B_i : X^i \times X_i \multimap Z$ and $P_i, Q_i : X^i \times X_i \multimap Z_i$ be multimaps satisfying the following conditions*:

(1) *For all $x \in X$, $A_i(x) \not\subseteq \operatorname{int} C_i$ implies $B_i(x) \not\subseteq \operatorname{int} C_i$ and $P_i(x) \not\subseteq -\operatorname{int} C_i$ implies $Q_i(x) \not\subseteq -\operatorname{int} C_i$;*
(2) *For all $x^i \in X^i$, the set $\{y_i \in X_i \mid A_i(x^i, y_i) \subseteq \operatorname{int} C_i\}$ is convex and for all $y_i \in X_i$ the set $\{x^i \in X^i \mid P_i(x^i, y_i) \subseteq -\operatorname{int} C_i\}$ is convex;*
(3) *The multimap $y_i \in X_i \to \{x^i \in X^i \mid B_i(x^i, y_i) \not\subseteq \operatorname{int} C_i\}$ is transfer compactly closed on X_i and the multimap $x^i \in X^i \to \{y_i \in X_i \mid Q_i(x^i, y_i) \not\subseteq -\operatorname{int} C_i\}$ is transfer compactly closed on X^i;*
(4) *For all $i \in I$ and $x^i \in X^i$, there exists $y_i \in X_i$ such that $B_i(x^i, y_i) \subseteq \operatorname{int} C_i$;*
(5) *If X^i is not compact, there exist a nonempty compact subset $K(i)$ of X^i and a compact convex subset D_i of X_i such that for each $x^i \in X^i \backslash K(i)$, there exists $y_i \in X_i$ such that $x^i \in \operatorname{cint}\{u^i \in X^i \mid B_i(u^i, y_i) \subseteq \operatorname{int} C_i\}$;*
(6) *For all $i \in I$, $y_i \in X_i$, there exists $x^i \in X^i$ such that $Q_i(x^i, y_i) \subseteq -\operatorname{int} C_i$;*
(7) *If X_i is not compact, there exist a nonempty compact subset $M(i)$ of X_i and a compact convex subset L^i of X^i such that for each $y_i \in X_i \backslash M(i)$, there exists $x^i \in L_i$ such that $y_i \in \operatorname{cint}\{u_i \in X_i \mid Q_i(x^i, u_i) \subseteq -\operatorname{int} C_i\}$.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X = \prod_{i\in I} X_i$, $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that $A_i(\bar{x}^i, \bar{y}_i) \subseteq \operatorname{int} C_i$ and $P_i(\bar{x}^i, \bar{y}_i) \subseteq -\operatorname{int} C_i$ for all $i \in I$.

Proof. For each $i \in I$, we define $S_i, T_i : X^i \multimap X_i$ by $S_i(x^i) = \{y_i \in X_i \mid B_i(x^i, y_i) \subseteq \operatorname{int} C_i\}$, $T_i(x^i) = \{y_i \in X_i \mid A_i(x^i, y_i) \subseteq \operatorname{int} C_i\}$, and $F_i, G_i : X_i \multimap X^i$ by $F_i(y_i) = \{x^i \in X^i \mid Q_i(x^i, y_i) \subseteq -\operatorname{int} C_i\}$, $G_i(y_i) = \{x^i \in X^i \mid P_i(x^i, y_i) \subseteq -\operatorname{int} C_i\}$.

(1) and (2) imply that $\operatorname{co}(S_i(x^i)) \subseteq T_i(x^i)$ for all $x^i \in X^i$ and $\operatorname{co}(F_i(y_i)) \subseteq G_i(y_i)$ for all $y_i \in X_i$.

(3) implies that S_i^- is transfer compactly open on X_i and F_i^- is transfer compactly open on X^i.

(4) implies that for all $x^i \in X^i$, $S_i(x^i)$ is nonempty. Therefore, $S_i(x^i)$ is nonempty on each compact subset of X^i and condition (ii) of Theorem 3.1 is satisfied.

For each finite subset N_i of $\langle X_i \rangle$ and R_i of $\langle X^i \rangle$, let $L_{N_i} = \mathrm{co}\{C_i \cup N_i\}$ and $L_{R_i} = \mathrm{co}\{D_i \cup R_i\}$. Then L_{N_i} is a compact convex subset of X_i containing N_i and L_{R_i} is a compact convex subset of X^i containing R_i.

(5) implies that $X^i \setminus K(i) \subset \bigcup\{\mathrm{cint}\, S_i^-(y_i):\ y_i \in L_{N_i}\}$.

(6) implies that for all $y_i \in X_i$, $F_i(y_i)$ is nonempty. This together with (3) imply condition (v) of Theorem 3.1.

(7) implies that $X_i \setminus K(i) \subset \bigcup\{\mathrm{cint}\, F_i^-(x^i):\ x^i \in L_{R_i}\}$.

Then, by Theorem 4.1 there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that

$$\bar{y}_i \in T_i(\bar{x}^i) \quad \text{and} \quad \bar{x}^i \in G_i(\bar{y}_i).$$

Therefore, $A_i(\bar{x}^i, \bar{y}_i) \subseteq \mathrm{int}\, C_i$ and $P_i(\bar{x}^i, \bar{y}_i) \subseteq -\mathrm{int}\, C_i$. □

As a simple consequence of Theorem 4.1 we have the following theorem which establishes the existence of solution for a system of inequalities.

Theorem 4.2. *Let I be an index set and for each $i \in I$, X_i be a nonempty convex subset of t.v.s. E_i, $f_i, g_i : X^i \times X_i \to \mathbb{R}$, $p_i, q_i : X^i \times X_i \to \mathbb{R}$ be functions, and $\{a_i\}_{i\in I}$ and $\{b_i\}_{i\in I}$ be families of real numbers. Suppose that for each $i \in I$, the following conditions hold*:

(1) *$g_i(x) \leqslant f_i(x)$ and $p_i(x) \leqslant q_i(x)$ for all $x \in X$;*
(2) *For each $x^i \in X^i$, $x_i \to f_i(x^i, x_i)$ is quasiconcave on X_i and for each $x_i \in X_i$, $x^i \to p_i(x^i, x_i)$ is quasiconvex on X^i;*
(3) *For each $x_i \in X_i$, $x^i \to g_i(x^i, x_i)$ is transfer compactly l.s.c. on X^i and for each $x^i \in X^i$, $x_i \to q_i(x^i, x_i)$ is transfer compactly u.s.c. on X_i;*
(4) *For $x^i \in X^i$, there exist $x_i \in X_i$ such that $g_i(x^i, x_i) > a_i$;*
(5) *If X^i is not compact, there exist a nonempty compact subset $K(i)$ of X^i and a nonempty compact convex subset D_i of X_i such that for each $x^i \in X^i \setminus K(i)$, there exists $y_i \in D_i$ such that $x^i \in \mathrm{cint}\{u^i \in X^i \mid g_i(u^i, y_i) > a_i\}$;*
(6) *For each $x_i \in X_i$, there exists $x^i \in X^i$ such that $q_i(x^i, x_i) < b_i$;*
(7) *If X_i is not compact, there exist a nonempty compact subset $M(i)$ of X_i and a nonempty compact convex subset L^i of X^i such that for each $y_i \in X_i \setminus M(i)$, there exists $x^i \in L^i$ such that $y_i \in \mathrm{cint}\{u_i \in X_i \mid q_i(x^i, u_i) < b_i\}$.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that

$$f_i(\bar{x}_i, \bar{y}_i) > a_i \quad \text{and} \quad p_i(\bar{x}_i, \bar{y}_i) < b_i \quad \text{for all } i \in I.$$

Proof. For each $i \in I$, let $C_i = [0, \infty)$ and $A_i, B_i, P_i, Q_i : X^i \times X_i \to \mathbb{R}$ be defined by

$$A_i(x^i, x_i) = f_i(x^i, x_i) - a_i, \qquad B_i(x^i, x_i) = g_i(x^i, x_i) - a_i,$$

and

$$P_i(x^i, x_i) = p_i(x^i, x_i) - b_i, \qquad Q_i(x^i, x_i) = q_i(x^i, x_i) - b_i.$$

Then all the conditions of Theorem 4.1 are satisfied. It follows from Theorem 4.1 that there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that $f_i(\bar{x}^i, \bar{y}_i) > a_i$ and $p_i(\bar{x}^i, \bar{y}_i) < b_i$ for all $i \in I$. □

Remark 4. (a) In Theorem 4.2, if for each $i \in I$, we let

$$S_i(x^i) = \{y_i \in X_i \mid g_i(x^i, y_i) - a_i > 0\},$$
$$F_i(x^i) = \{y_i \in X_i \mid f_i(x^i, y_i) - a_i > 0\},$$
$$H_i(x_i) = \{x^i \in X^i \mid q_i(x^i, x_i) - b_i < 0\},$$
$$T_i(x_i) = \{x^i \in X^i \mid p_i(x^i, x_i) - b_i < 0\}.$$

Then all the conditions of Theorem 3.1 are satisfied and the conclusion of Theorem 4.2 follows from Theorem 3.1.

(b) If I is a singleton, then Theorem 4.2 reduces to Theorem 5.6 in [3].

By Theorem 4.2, we have the following system minimax theorem.

Theorem 4.3. *For each $i \in I$, let X_i be a nonempty convex subset of t.v.s. E_i, $f_i, g_i : X^i \times X_i \to \mathbb{R}$, $p_i, q_i : X^i \times X_i \to \mathbb{R}$ be functions such that*

(1) $g_i(x) \leqslant f_i(x) \leqslant p_i(x) \leqslant q_i(x)$ *for all* $x \in X$;
(2) *For each $x^i \in X^i$, $x_i \to f_i(x^i, x_i)$ is quasiconcave on X_i and for each $x_i \in X_i$, $x^i \to p_i(x^i, x_i)$ is quasiconvex on X^i;*
(3) *For each $x_i \in X_i$, $x^i \to g_i(x^i, x_i)$ is transfer compactly l.s.c. on X^i and for each $x^i \in X^i$, $x_i \to q_i(x^i, x_i)$ is transfer compactly u.s.c. on X_i;*
(4) *If X^i is not compact, there exist a nonempty compact subset $K(i)$ of X^i and a nonempty compact convex subset D_i of X_i such that for each $x^i \in X^i \setminus K(i)$, there exists $y_i \in D_i$ such that*
$$x^i \in \operatorname{cint}\Big\{u^i \in X^i \mid g_i(u^i, y_i) \geqslant \inf_{u^i \in X^i} \sup_{u_i \in X_i} g_i(u^i, u_i)\Big\};$$
(5) *If X_i is not compact, there exist a nonempty compact subset $M(i)$ of X_i and a nonempty compact convex subset L_i of X^i such that for each $y_i \in X_i \setminus M(i)$, there exists $x^i \in L_i$ such that*
$$y_i \in \operatorname{cint}\Big\{u_i \in X_i \mid q_i(x^i, u_i) \geqslant \inf_{u^i \in X^i} \sup_{u_i \in X_i} q_i(u^i, u_i)\Big\}.$$

Then

$$\inf_{u^i \in X^i} \sup_{u_i \in X_i} g_i(u^i, u_i) \leqslant \sup_{u_i \in X_i} \inf_{u^i \in X^i} q_i(u^i, u_i) \quad \text{for all } i \in I.$$

Proof. Let $\varepsilon > 0$ and for each $i \in I$, let

$$a_i = \inf_{u^i \in X^i} \sup_{u_i \in X_i} g_i(u^i, u_i) - \varepsilon \quad \text{and} \quad b_i = \sup_{u_i \in X_i} \inf_{u^i \in X^i} q_i(u^i, u_i) + \varepsilon.$$

Then for each $x^i \in X^i$, there exists $x_i \in X_i$ such that $g_i(x^i, x_i) > a_i$ and for each $x_i \in X_i$, there exists $x^i \in X^i$ such that $q_i(x^i, x_i) < b_i$. Therefore, all the conditions of Theorem 4.2 are satisfied.

It follows from Theorem 4.2 that there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that for all $i \in I$,

$$f_i(\bar{x}^i, \bar{y}_i) > \inf_{u^i\in X^i} \sup_{u_i\in X_i} g_i(u^i, u_i) - \varepsilon \quad \text{and} \quad p_i(\bar{x}^i, \bar{y}_i) < \sup_{u_i\in X_i} \inf_{u^i\in X^i} q_i(u^i, u_i) + \varepsilon.$$

Since $p_i(\bar{x}^i, \bar{y}_i) \geqslant f_i(\bar{x}^i, \bar{y}_i)$,

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} q_i(u^i, u_i) + \varepsilon > \inf_{u^i\in X^i} \sup_{u_i\in X_i} g_i(u^i, u_i) - \varepsilon,$$

since ε is arbitrary positive number,

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} q_i(u^i, u_i) \geqslant \inf_{u^i\in X^i} \sup_{u_i\in X_i} g_i(u^i, u_i) \quad \text{for all } i \in I. \qquad \Box$$

If we let $p_i = f_i = q_i = g_i$ in Theorem 4.3, we have the following corollary.

Remark 5. Theorem 4.3 is different from Theorem 4b in [5].

Corollary 4.4. *In Theorem* 4.3, *if for all $i \in I$, $f_i = g_i = p_i = q_i$, then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that for all $i \in I$,*

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} f_i(u^i, u_i) = \inf_{u^i\in X^i} \sup_{u_i\in X_i} f_i(u^i, u_i).$$

Proof. By Theorem 4.3, we see that for all $i \in I$,

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} f_i(u^i, u_i) \geqslant \inf_{u^i\in X^i} \sup_{u_i\in X_i} f_i(u^i, u_i).$$

Since for all $i \in I$, we have

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} f_i(u^i, u_i) \leqslant \inf_{u^i\in X^i} \sup_{u_i\in X_i} f_i(u^i, u_i),$$

it follows that

$$\sup_{u_i\in X_i} \inf_{u^i\in X^i} f_i(u^i, u_i) = \inf_{u^i\in X^i} \sup_{u_i\in X_i} f_i(u^i, u_i). \qquad \Box$$

Theorem 4.5. *Let I be an index set. For each $i \in I$, let X_i be a nonempty compact convex subset of t.v.s. E_i, $f_i : X^i \times X_i \to \mathbb{R}$ be a function satisfying the following conditions*:

(i) *For each $x^i \in X_i$, $x_i \to f_i(x^i, x_i)$ is quasiconcave and u.s.c. on X_i*;
(ii) *For each $x_i \in X^i$, $x^i \to f_i(x^i, x_i)$ is quasiconvex and l.s.c. on X^i.*

Then there exist $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in X$ such that for all $i \in I$,

$$\min_{u^i\in X^i} \max_{u_i\in X_i} f_i(u^i, u_i) = f_i(\bar{x}^i, \bar{y}_i) = \max_{u_i\in X_i} \min_{u^i\in X^i} f_i(u^i, u_i).$$

Proof. Theorem 4.5 follows immediately from the compactness of X_i and Remark 5. $\Box$

Remark 6. If I is a singleton, then Theorem 4.5 reduces to the Sion's mimimax theorem [16].

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