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Solutions of system of generalized vector quasi-equilibrium problems in locally G-convex uniform spaces

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Abstract

In this paper we establish a collectively fixed point theorem and an equilibrium existence theorem for generalized games in product locally G-convex uniform spaces. As applications, some new existence theorems of solutions for the system of generalized vector quasi-equilibrium problems are derived in product locally G-convex uniform spaces. These theorems are new and generalize some known results in the literature.

Keywords: Collectively fixed point; Generalized game; System of generalized vector quasi-equilibrium problems; Locally G-convex uniform space

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1. Introduction and preliminaries

The vector variational inequality problem was first introduced and studied by Giannessi [22] in finite dimensional Euclidean spaces. Since then, such problem has been extended and generalized by many authors in various different directions. We noticed that vector variational inequalities and its various generalizations have extensive and important applications in vector optimization, optimal control, mathematical programming, operations research and equilibrium problem of economics, etc. Inspired and motivated by above applications, various generalized vector variational inequality problems and generalized vector equilibrium problems have become important developed directions of the classical vector variational inequality theory, for example, see [2,8,11,12,23,26,27,30,31] and references therein.

Pang [32] has shown that a variety of equilibrium models, for example, the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium problem and the general equilibrium programming problem can be uniformly modelled as a variational inequality defined on the product sets. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve. The method of decomposition was also used by Zhu and Marcotte [39], and Cohen and Chaplais [10] to solve a variational inequality problem defined on a set of inequality constraints. Motivated and inspired by the above development, Ansari and Yao [3,4], Ansari et al. [5,6], and Ding [13,14] introduced and studied the system of vector equilibrium problems. Some existence theorems of solutions for the system of vector equilibrium problems are established in topological vector spaces and G-convex spaces, respectively.

On the other hand, the quasi-equilibrium problems, the generalized quasi-equilibrium problems and their applications have also been studied extensively by many authors, for example, see [9,15,16,18,19,28,29,33].

Following the trend of the above research fields, we will introduce and study some new classes of system of generalized vector quasi-equilibrium problems on a product space of G-convex spaces in this paper.

Let I be a finite or infinite index set, $\{X_i\}_{i\in I}$ and $\{Y_i\}_{i\in I}$ be two families of topological spaces, and $\{Z_i\}_{i\in I}$ be a family of nonempty sets. Let $X=\prod_{i\in I} X_i$ and $Y=\prod_{i\in I} Y_i$. For each $i\in I$, let $A_i : X\to 2^{X_i}$, $F_i : X\times X_i\to 2^{Z_i}$, $T_i : X\to 2^{Y_i}$, $C_i : X\to 2^{Z_i}$ and $\varphi_i : X\times Y_i\times X_i\to 2^{Z_i}$ be set-valued mappings.

A system of generalized vector quasi-equilibrium problems of type (I) (SGVQEP(I)) is to find $\hat{x}\in X$ such that for each $i\in I$,

$$\hat{x}_i\in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x},z_i)\nsubseteq C_i(\hat{x}) \quad \forall z_i\in A_i(\hat{x}). \tag{1.1}$$

A system of generalized vector quasi-equilibrium problems of type (II) (SGVQEP(II)) is to find $\hat{x}\in X$ and $\hat{y}\in Y$ such that for each $i\in I$,

$$\hat{x}_i\in A_i(\hat{x}),\quad \hat{y}_i\in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x},\hat{y}_i,z_i)\nsubseteq C_i(\hat{x}) \quad \forall z_i\in A_i(\hat{x}). \tag{1.2}$$

A system of generalized vector quasi-equilibrium problems of type (III) (SGVQEP(III)) is to find $\hat{x}\in X$ such that for each $i\in I$,

$$\hat{x}_i\in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x},z_i)\subseteq C_i(\hat{x}) \quad \forall z_i\in A_i(\hat{x}). \tag{1.3}$$

A system of generalized vector quasi-equilibrium problems of type (IV) (SGVQEP(IV)) is to find $\hat{x} \in X$ and $\hat{y} \in Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}). \tag{1.4}$$

If $Z_i = [-\infty, +\infty]$ and $C_i(x) = [-\infty, 0]$ for all $i \in I$ and $x \in X$, and $F_i(x, z_i) = \varphi_i(x, z_i)$ for each $(x, z_i) \in X \times X_i$ is a single-valued function, then the SGVQEP(I) reduces to the following system of generalized quasi-equilibrium problems of type (V) (SGQEP(V)) which is to find $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, z_i) > 0 \quad \forall z_i \in A_i(\hat{x}). \tag{1.5}$$

When I is a singleton, problem (1.5) has been introduced and studied by many authors. For example, see [9,15,16,18,19,28,29,33].

If $Z_i = [-\infty, +\infty]$ and $C_i(x) = [-\infty, 0)$ for all $i \in I$ and $x \in X$ and $\varphi_i(x, y_i, z_i)$ is a single-valued function, then the SGVQEP(II) reduces to the following system of generalized quasi-equilibrium problems of type (VI) (SGQEP(VI)) which is to find $\hat{x} \in X$ and $\hat{y} \in Y$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) < 0 \quad \forall z_i \in A_i(\hat{x}). \tag{1.6}$$

When I is a singleton and $X = Y$, problem (1.6) has been introduced and studied by Ding and Park [19], Lin and Park [28], Lin and Yu [29], and Chen et al. [9] and Park [33], respectively.

If $A_i(x) = X_i$ for each $i \in I$ and $x \in X$, then the SGVQEP(I) reduces to the following system of generalized vector equilibrium problems of type (VII) (SGVEP(VII)) which is to find $\hat{x} \in X$ such that for each $i \in I$,

$$F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in X_i. \tag{1.7}$$

The SGVEP(VII) and its special cases have been introduced and studied by Ansari et al. [5,6] in topological vector spaces. When I is a singleton, problem (1.7) and its special cases have been extensively studied by many authors. For example, see [2,8,11,12,23,26, 27,30,31] and references therein.

If $A_i(x) = X_i$ for each $i \in I$ and $x \in X$, then the SGVQEP(II) reduces to the following system of generalized vector equilibrium problems of type (VIII) (SGVEP(VIII)) which is to find $\hat{x} \in X$ and $\hat{y} \in Y$ such that for each $i \in I$,

$$\hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in X_i. \tag{1.8}$$

Obviously, SGVQEP(I) and SGVQEP(II) include problems (1.3)–(1.8) as special cases. For appropriate choices of the spaces X_i, Y_i, Z_i and the mappings A_i, T_i, C_i and φ_i, it is easy to see that SGVQEP(I) and SGVQEP(II) include a number of extensions and generalizations of generalized (vector) equilibrium problems, generalized (vector) variational inequality problems, generalized (vector) quasi-equilibrium problems and generalized (vector) quasi-variational inequality problems as special cases. For example, see [2–6, 8–16,18,19,21–23,26–33,39] and references therein.

In this paper, we will establish a collectively fixed point theorem and an equilibrium existence theorem for generalized games in product locally G-convex uniform spaces. As applications, some new existence theorems of solutions for SGVQEP(I) and SGVQEP(II) are derived in product locally G-convex uniform spaces. Let us first recall the following preliminaries which will be needed in the sequel.

For a set X, we will denote by 2^X and $\mathcal{F}(X)$ the family of all subsets of X and the family of all nonempty finite subsets of X, respectively. For $A \in \mathcal{F}(X)$, we denote by $|A|$ the cardinality of A. Let Δ_n be the standard n-dimensional simplex with vertices $e_0, e_1, \ldots, e_n$. If J is a nonempty subset of $\{0, 1, \ldots, n\}$, we denote by Δ_J the convex hull of the vertices $\{e_j\colon j \in J\}$. The notion of a generalized convex (or G-convex) space was introduced under an extra isotonic condition by Park and Kim [34]. Recently Park [35] gave the following definition of a G-convex space by removing the extra condition.

A G-convex space $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D and a set-valued mapping $\Gamma : \mathcal{F}(D) \to 2^X \setminus \{\emptyset\}$ such that for each $A = \{a_0, a_1, \ldots, a_n\} \in \mathcal{F}(D)$ with $|A| = n + 1$, there exists a continuous mapping $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subseteq \{0, 1, \ldots, n\}$ implies $\phi_A(\Delta_J) \subseteq \Gamma(\{a_j\colon j \in J\})$, where $\Delta_J = \mathrm{co}\{e_j\colon j \in J\}$, the convex hull of the set $\{e_j\colon j \in J\}$. When $D = X$, we will write $(X, X; \Gamma) = (X, \Gamma)$. In the case of $D \subseteq X$, a subset C of $(X, D; \Gamma)$ is said to be Γ-convex if for each $A \in \mathcal{F}(D \cap C)$, $\Gamma(A) \subseteq C$.

A locally G-convex uniform space (also see [35]) is a G-convex space $(X, D; \Gamma)$ such that

(1) X is a separated uniform space with the basis β for symmetric entourages;
(2) D is a dense subset of X; and
(3) for each $V \in \beta$ and each $x \in X$, the set $V[x] = \{y \in X\colon (x, y) \in V\}$ is Γ-convex.

G-convex spaces include the convex subsets of a topological vector space, H-spaces (see Horvath [24,25]) and many topological spaces with abstract convexity structure as special cases; see [34–36]. The notion of locally G-convex uniform spaces generalizes the notions of locally convex H-spaces and locally G-convex spaces introduced by Wu and Li [37] and Yuan [38], respectively.

Let $(X, D; \Gamma)$ be a G-convex space and Z be a nonempty set. Let $F : X \to 2^Z$ and $C : X \to 2^Z$ be set-valued mappings. F is said to be G-quasi-convex (respectively, G-quasi-concave) with respect to C if the set $\{x \in X\colon F(x) \subseteq C(x)\}$ (respectively, $\{x \in X\colon F(x) \nsubseteq C(x)\}$) is G-convex.

Let X and Y be both topological spaces. A set-valued mapping $G : X \to 2^Y$ is said to be compact if $G(X)$ is contained in some compact subset of Y. G is said to be upper semicontinuous (u.s.c.) (respectively, lower semicontinuous (l.s.c.)) on X if for each $x \in X$ and for each open set U of Y, the set $\{x \in X\colon G(x) \subseteq U\}$ (respectively, $\{x \in X\colon F(x) \cap U \neq \emptyset\}$) is open in X. If $S, T : X \to 2^Y$ are set-valued mappings, then $(S \cap T) : X \to 2^Y$ is the set-valued mapping defined by $(S \cap T)(x) = T(x) \cap S(x)$ for each $x \in X$.

In order to prove the main result (Theorem 1.2) of this section, we need the following results.

Theorem 1.1 [17, Theorem 3.1]. *Let $(X_i, D_i; \Gamma_i)_{i \in I}$ be a family of locally G-convex uniform spaces with each X_i having the basis β_i of symmetric entourages. For each $i \in I$, let $G_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ be an upper semicontinuous compact set-valued mapping with* *nonempty closed Γ_i-convex values. Then there exists a point $\hat{x} = (\hat{x}_i)_{i \in I} \in X$ such that $\hat{x}_i \in G_i(\hat{x})$ for each $i \in I$.*

Lemma 1.1 [1, Theorem 14.18]. *Let X, Y be topological spaces and $\varphi_i : X \to 2^Y$ be a set-valued mapping. Then the following statements are equivalent*:

(i) *φ is lower semicontinuous at a point $x \in X$,*
(ii) *if $x_\alpha \to x$, then for each $y \in \varphi(x)$ there exists a subnet $\{\alpha_\lambda\}_{\lambda \in \Lambda}$ of the index set $\{\alpha\}$ and elements $y_\lambda \in \varphi(x_{\alpha_\lambda})$ for each $\lambda \in \Lambda$ such that $y_\lambda \to y$.*

Lemma 1.2 [38, Lemma 4.7.3, p. 301]. *Let X and Y be two topological spaces and A be a closed* (*respectively, open*) *subset of X. Suppose $F_1 : X \to 2^Y$ and $F_2 : A \to 2^Y$ are both l.s.c.* (*respectively, u.s.c.*) *such that $F_2(x) \subseteq F_1(x)$ for each $x \in A$. Then the mapping $F : X \to 2^Y$ defined by*

$$F(x) = \begin{cases} F_2(x), & \text{if } x \in A, \\ F_1(x), & \text{if } x \notin A, \end{cases}$$

is also l.s.c. (*respectively, u.s.c.*).

Now we describe a generalized game $\mathcal{E} = (X_i, A_i, P_i)_{i \in I}$, where I is a finite or infinite set of agents; for each $i \in I$, X_i is a the strategy set (or commodity space) of ith agent; $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ is the constrained correspondence (set-valued mapping) and $P_i : X \to 2^{X_i}$ is the preference correspondence. A point $\hat{x} \in X$ is called an equilibrium point of the generalized game $\mathcal{E}$ if $\hat{x}_i \in A_i(\hat{x})$ and $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ for each $i \in I$.

Theorem 1.2. *Let $((X_i, D_i; \Gamma_i), A_i, P_i)_{i \in I}$ be a generalized game such that for each $i \in I$,*

(i) *$(X_i, D_i; \Gamma_i)$ is a locally G-convex uniform space,*
(ii) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values on X,*
(iii) *the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X,*
(iv) *$A_i \cap P_i : E \subseteq X \to 2^{X_i}$ is u.s.c. with closed Γ_i-convex values,*
(v) *for each $x \in X$, $x_i \notin A_i(x) \cap P_i(x)$.*

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

Proof. For each $i \in I$, define a set-valued mapping $T_i : X \to 2^{X_i}$ by

$$T_i(x) = \begin{cases} A_i(x) \cap P_i(x), & \text{if } x \in E_i, \\ A_i(x), & \text{if } x \notin E_i. \end{cases}$$

From conditions (ii)–(iv) and Lemma 1.2 it follows that for each $i \in I$, T_i is an u.s.c. compact mapping with nonempty closed Γ-convex values. By Theorem 1.1, there exists a point $\hat{x} \in X$ such that for each $i \in I$, $\hat{x}_i \in T_i(\hat{x})$. If for some $i \in I$, $\hat{x}_i \in E_i$, then we have

$\hat{x}_i \in A_i(\hat{x}) \cap P_i(\hat{x})$ which contradicts the condition (v). Hence we conclude that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset,$$

i.e., $\hat{x}$ is an equilibrium point of the generalized game $\mathcal{E}$. □

2. Existence of solutions for SGVQEP

In this section by using Theorem 1.2, we shall establish some new existence theorems of solutions for SGVQEP(I) and SGVQEP(II), respectively, in locally G-convex uniform spaces.

The following result is a variant of Theorem 3.1 of Ding and Park [20].

Lemma 2.1. *Let X be a normal space and $(Y, D; \Gamma)$ be a G-convex space. Let $T : X \to 2^Y$ be a set-valued mapping satisfying the following conditions*:

(i) *for each $x \in X$, $T(x)$ is Γ-convex,*
(ii) *there exists $M = \{y_0, y_1, \dots, y_n\} \in \mathcal{F}(D)$ such that $X = \bigcup_{y \in M} \operatorname{int} T^{-1}(y)$.*

Then there exists a continuous selection $f : X \to Y$ of T such that $f = \varphi_M \circ \psi$, where $\varphi_M : \Delta_n \to Y$ and $\psi : X \to \Delta_n$ are both continuous.

Proof. Since Y is a G-convex space and $M = \{y_0, y_1, \dots, y_n\} \in \mathcal{F}(D)$ by (ii), there exists a continuous mapping $\varphi_M : \Delta_n \to \Gamma(M)$ such that

$$\varphi_M(\Delta_J) \subseteq \Gamma(B), \quad \forall B \in \langle M \rangle,\ |B| = |J| + 1. \tag{2.1}$$

Since X is normal and $X = \bigcup_{i=0}^n \operatorname{int} T^{-1}(y_i)$ by (ii), there exists a continuous partition of unity $\{\psi_i\}_{i=0}^n$ subordinated to the open covering $\{\operatorname{int} T^{-1}(y_i)\}_{i=0}^n$ such that for each $i \in \{0, 1, \dots, n\}$ and $x \in X$, we have

$$\psi_i(x) \neq 0 \quad \Leftrightarrow \quad x \in \operatorname{int} T^{-1}(y_i) \quad \Rightarrow \quad y_i \in T(x). \tag{2.2}$$

Define a mapping $\psi : X \to \Delta_n$ by $\psi(x) = \sum_{i=0}^n \psi_i(x) e_i$. Then ψ is continuous and for each $x \in X$, $\psi(x) = \sum_{j \in J(x)} \psi_j(x) e_j \in \Delta_{J(x)}$, where $J(x) = \{j \in \{0, 1, \dots, n\}$: $\psi_j(x) \neq 0\}$. By (2.2), we have $\{y_j\colon J(x)\} \in \mathcal{F}(T(x))$. From (2.1) and the condition (i) we obtain

$$f(x) = (\varphi_M \circ \psi)(x) \in \varphi_M(\Delta_{J(x)}) \subseteq \Gamma\big(\{y_i\colon j \in J(x)\}\big) \subseteq T(x).$$

This shows that $f = \phi \circ \psi$ continuous selection of T. □

Theorem 2.1. *Let $(X_i, D_i; \Gamma_i)_{i \in I}$ be a family of locally G-convex uniform spaces and $\{Z_i\}_{i \in I}$ be a family of nonempty sets. For each $i \in I$, let $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}$, $F_i : X \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be three set-valued mappings such that for each $i \in I$,*

(i) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*

(ii) *the set* $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X *where the mapping* $P_i : X \to 2^{X_i}$ *is defined by* $P_i(x) = \{z_i \in X_i\colon F_i(x, z_i) \subseteq C_i(x)\}$,

(iii) *the mapping* $A_i \cap P_i : E_i \subseteq X \to 2^{X_i}$ *is u.s.c. with nonempty closed* Γ_i*-convex values,*

(iv) *for each* $x \in X$, $x_i \notin A_i(x) \cap P_i(x)$.

Then there exists $\hat{x} \in X$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \textit{and} \quad F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Proof. For each $i \in I$, define a set-valued mapping $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \big\{z_i \in X_i\colon F_i(x, z_i) \subseteq C_i(x)\big\} \quad \forall x \in X.$$

It is easy to check that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

It follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}),$$

i.e., $\hat{x}$ is a solution of the SGVQEP(I). □

Theorem 2.2. *Let* $(X_i, D_i; \Gamma_i)$ *be a family of locally* G*-convex uniform spaces and* $\{Z_i\}_{i\in I}$ *be a family of topological spaces. Let* $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$, $F_i : X \times X_i \to 2^{Z_i}$ *and* $C_i : X \to 2^{Z_i}$ *be three set-valued mappings. Suppose that for each* $i \in I$,

(i) A_i *is an u.s.c. compact mapping with nonempty closed* Γ_i*-convex values,*

(ii) *the set* $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X *where the mapping* $P_i : X \to 2^{X_i}$ *is defined by* $P_i(x) = \{z_i \in X_i\colon F_i(x_i z_i) \subseteq C_i(x)\}$,

(iii) $F_i(x, z_i)$ *is l.s.c. on* $X \times X_i$,

(iv) *the mapping* C_i *has closed graph,*

(v) *for each* $x \in X$, $z_i \mapsto F(x, z_i)$ *is* G*-quasi-convex with respect to* C_i,

(vi) *for each* $x \in X$, $F(x, x_i) \nsubseteq C_i(x)$.

Then there exists $\hat{x} \in X$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \textit{and} \quad F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Proof. Define a set-valued mapping $P_i : X \to 2^{X_i}$ by

$$P_i(x) = \big\{z_i \in X_i\colon F_i(x, z_i) \subseteq C_i(x)\big\} \quad \forall x \in X.$$

We claim that P_i has closed graph. Indeed, let $\{(x_\alpha, z_{i,\alpha})\}$ be a net in $\mathrm{Gr}(P_i)$ and $(x_\alpha, z_{i,\alpha}) \to (x_0, z_{i,0})$. Then we have that $F_i(x_\alpha, z_{i,\alpha}\} \subseteq C_i(x_\alpha)$ for each α. If $F(x_0, z_{i,0}) \nsubseteq C_i(x_0)$, then there exists a point $u_{i,0} \in F(x_0, z_{i,0})$ such that $u_{i,0} \notin C_i(x_0)$. By the condition (iii) and Lemma 1.1, there exists a subnet $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ of $\{\alpha\}$ and $u_{i,\alpha_\lambda} \in F_i(x_{\alpha_\lambda}, z_{i,\alpha_\lambda})$ such that $u_{i,\alpha_\lambda} \to u_{i,0}$. Since $u_{i,\alpha_\lambda} \in F_i(x_{\alpha_\lambda}, z_{i,\alpha_\lambda}) \subseteq C_i(x_{\alpha_\lambda})$ for each $\lambda \in \Lambda$ and C_i has closed graph, we must have $u_{i,0} \in C_i(x_0)$ which is a contradiction. Hence we have

$F_i(x_0, z_{i,0}) \subseteq C_i(x_0)$. So the mapping P_i has closed graph. By Theorem 3.1.8 of Aubin–Ekeland [7], $A_i \cap P_i : E_i \subseteq X \to 2^{X_i}$ is an u.s.c. mapping with nonempty closed values. By (v), P_i has Γ_i-convex values and hence $(A_i \cap P_i)(x)$ is Γ_i-convex for each $x \in E$ by (i). The condition (iii) of Theorem 2.1 is clearly satisfied. By (vi) for each $x \in X$, $x_i \notin P_i(x)$. Hence $x_i \notin A_i(x) \cap P_i(x)$ and the condition (iv) of Theorem 2.1 is also satisfied. The conclusion of Theorem 2.2 now follows from Theorem 2.1.

Corollary 2.1. *Let $(X_i, D_i; \Gamma_i)$ be a family of locally G-convex uniform spaces. Let $A_i : X \to 2^{X_i}$ be a set-valued mapping and $\varphi_i : X \times X_i \to [-\infty, +\infty]$ be a single-valued continuous function such that for each $i \in I$,*

(i) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
(ii) *the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon \varphi(x, z_i) \leqslant 0\}$,*
(iii) *for each $x \in X$, the set $\{z_i \in X_i\colon \varphi_i(x, z_i) \leqslant 0\}$ is Γ_i-convex,*
(iv) *for each $x \in X$, $\varphi_i(x, x_i) > 0$.*

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \textit{and} \quad \varphi_i(\hat{x}, z_i) > 0 \quad \forall z_i \in A_i(\hat{x}).$$

Proof. Let $Z_i = [-\infty, +\infty]$, $C_i(x) = [-\infty, 0]$ for each $x \in X$ and $F_i(x, z_i) = \{\varphi(x, z_i)\}$ for all $(x, z_i) \in X \times X_i$. It is easy to check that all conditions of Theorem 2.2 are satisfied. The conclusion of Corollary 2.2 follows from Theorem 2.2. □

Theorem 2.3. *Let $(X_i, D_i; \Gamma_i)$ be a family of locally G-convex uniform space and $\{Z_i\}_{i \in I}$ be a family of topological spaces. Let $A_i : X \to 2^{X_i}$, $F_i : X \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be three set-valued mappings such that for each $i \in I$,*

(i) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
(ii) *the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon F_i(x, z_i) \nsubseteq C_i(x)\}$,*
(iii) *$F_i(x, z_i)$ is an u.s.c. compact mapping with closed values,*
(iv) *the mapping C_i has open graph,*
(v) *for each $x \in X$, $z_i \mapsto F_i(x, z_i)$ is G-quasi-concave with respect to C_i,*
(vi) *for each $x \in X$, $F_i(x, x_i) \subseteq C_i(x)$.*

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \textit{and} \quad F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Proof. Define set-valued mappings $P, H : X \to 2^{X_i}$ by

$$P_i(x) = \bigl\{z_i \in X_i\colon F_i(x, z_i) \nsubseteq C_i(x)\bigr\} \quad \forall x \in X,$$

and

$$H_i(x) = A_i(x) \cap P_i(x) = \bigl\{z_i \in A_i(x)\colon F_i(x, z_i) \cap \bigl(Z_i \setminus C_i(x)\bigr) \neq \emptyset\bigr\} \quad \forall x \in X,$$

respectively. Since A_i is a compact mapping, H_i is also a compact mapping. We claim that H_i has closed graph. Indeed, let $\{(x_\alpha, z_{i,\alpha})\}_{\alpha\in I}$ be a net in $\mathrm{Gr}(H_i)$ and $(x_\alpha, z_{i,\alpha}) \to (x_0, z_{i,0})$. Then we have $z_{i,\alpha} \in A_i(x_\alpha)$ and $F_i(x_\alpha, z_{i,\alpha}) \cap (Z_i \setminus C_i(x_\alpha)) \neq \emptyset$ for each $\alpha \in I$. Hence there exists $u_{i,\alpha} \in F_i(x_\alpha, z_{i,\alpha})$ such that $u_{i,\alpha} \in Z_i \setminus C_i(x_\alpha)$ for each $\alpha \in I$. Without loss of generality, by (iii) we can assume that $u_{i,\alpha} \to u_{i,0}$ and so $u_{i,0} \in F_i(x_0, z_{i,0})$. By (iv) the mapping $W_i : X \to 2^{Z_i}$ defined by $W_i(x) = Z_i \setminus C_i(x)$ has closed graph. It follows that $u_{i,0} \in W_i(x_0) = Z_i \setminus C_i(x_0)$ and $F_i(x_0, z_{i,0}) \cap (Z_i \setminus C_i(x_0)) \neq \emptyset$. By (i) we have $z_{i,0} \in A_i(x_0)$. Therefore $(x_0, z_{i,0}) \in \mathrm{Gr}(H_i)$ and the graph $\mathrm{Gr}(H_i)$ of H_i is closed. Hence $H_i = A_i \cap P_i$ is an u.s.c. compact mapping with nonempty closed values. By (i) and (v) $A_i \cap P_i$ has Γ_i-convex values. The condition (iii) of Theorem 2.1 is clearly satisfied. By (vi) for each $x \in X$, $x_i \notin P_i(x)$ and hence $x_i \notin A_i(x) \cap P_i(x)$, from which the condition (iv) of Theorem 2.1 is satisfied. Therefore by Theorem 2.1, there exists a point $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset.$$

It follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

The proof is now complete. □

Corollary 2.2. *Let $(X_i, D_i; \Gamma_i)$ be a family of locally G-convex uniform spaces. Let $A : X \to 2^{X_i}$ be a set-valued mapping and $\varphi_i : X \times X_i \to [-\infty, +\infty]$ be a function such that for each $i \in I$,*

(i) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
(ii) *the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon \varphi_i(x, z_i) \geqslant 0\}$,*
(iii) *$\varphi_i(x, y)$ is a continuous bounded function,*
(vi) *for each $x \in X$, the set $\{z_i \in X_i\colon \varphi_i(x, z_i) \geqslant 0\}$ is Γ_i-convex,*
(v) *for each $x \in X$, $\varphi_i(x, x_i) < 0$.*

Then there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \textit{and} \quad \varphi_i(\hat{x}, z_i) < 0 \quad \forall z_i \in A_i(\hat{x}).$$

Proof. Let $Z = [-\infty, +\infty]$, $C_i(x) = [-\infty, 0)$ for each $x \in X$ and $F_i(x, z_i) = \{\varphi_i(x, z_i)\}$ for all $(x, z_i) \in X \times X_i$. It is easy to check that all conditions of Theorem 2.3 are satisfied. The conclusion of Corollary 2.2 now follows from Theorem 2.3. □

Remark 2.1. Theorems 2.1–2.3 and Corollaries 2.1 and 2.2 are new results which are different from the corresponding results in [2–6,8–16,18,19,22,23,26–33,39] and our argument methods are also different.

Theorem 2.4. *Let $(X_i, D_i; \Gamma_i)$ be a family of locally G-convex uniform spaces such that $X = \prod_{i\in I} X_i$ is a normal space. Let $(Y_i, D_i'; \Gamma_i')_{i\in I}$ be a family of G-convex spaces and*

$\{Z_i\}_{i\in I}$ be a family of topological spaces. For each $i \in I$, let $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$, $T_i : X \to 2^{Y_i}$, $\varphi_i : X \times Y_i \times X_i \to 2^{Z_i}$ and $C_i : X \to 2^{Z_i}$ be set-valued mappings. Suppose that for each $i \in I$,

- (i) *T_i has nonempty Γ_i'-convex values,*
- (ii) *there exists $M_i = \{y_{i,0}, \dots, y_{i,n_i}\} \in \mathcal{F}(D_i')$ such that $X = \bigcup_{y_i\in M_i} \operatorname{int} T_i^{-1}(y_i)$,*
- (iii) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
- (iv) *for any continuous mapping $f_i : X \to Y_i$ the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon \varphi_i(x, f_i(x), z_i) \subseteq C_i(x)\}$,*
- (v) *$\varphi_i(x, y_i, z_i)$ is l.s.c. on $X \times Y_i \times X_i$,*
- (vi) *the mapping C_i has closed graph,*
- (vii) *for each $(x, y_i) \in X \times Y_i$, $z_i \mapsto \varphi(x, y_i, z_i)$ is G-quasi-convex with respect to C_i,*
- (viii) *for any continuous mapping $f_i : X \to Y_i$ and $x \in X$, $\varphi_i(x, f_i(x), x_i) \nsubseteq C_i(x)$.*

Then there exist $\hat{x} \in X$ and $\hat{y} \in Y = \prod_{i\in I} Y_i$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \textit{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Proof. By conditions (i), (ii) and Lemma 2.1 for each $i \in I$, T_i has a continuous selection $f_i : X \to Y_i$. For each $i \in I$, define a set-valued mapping $F_i : X \times X_i \to 2^{Z_i}$ by

$$F_i(x, z_i) = \varphi_i\big(x, f_i(x), z_i\big) \quad \forall (x, z_i) \in X \times X_i.$$

By conditions (iii)–(viii), it is easy to see that all conditions of Theorem 2.2 are satisfied. Hence by Theorem 2.2, there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x}, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

For each $i \in I$, let $\hat{y}_i = f_i(\hat{x})$. Then $\hat{y} \in Y = \prod_{i\in I} Y_i$. Note that f_i is a continuous selection of T_i. We get for each $i \in I$, $\hat{y}_i = f_i(\hat{x}) \in T_i(\hat{x})$ from which it follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \nsubseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Corollary 2.3. *Let $(X_i, D_i; \Gamma_i)$ be a family of locally G-convex uniform spaces such that $X = \prod_{i\in I} X_i$ is a normal space. Let $(Y_i, D_i'; \Gamma_i')_{i\in I}$ be a family of G-convex spaces. For each $i \in I$, let $A_i : X = \prod_{i\in I} X_i \to 2^{X_i}$, $T_i : X \to 2^{Y_i}$ and $C_i : X \to 2^{Z_i}$ be set-valued mappings. Let $\varphi_i : X \times Y_i \times X_i \to [-\infty, +\infty]$ be a continuous function. Suppose that for each $i \in I$,*

- (i) *T_i has nonempty Γ_i'-convex values,*
- (ii) *there exists $M_i = \{y_{i,0}, \dots, y_{i,n_i}\} \in \mathcal{F}(D_i')$ such that $X = \bigcup_{y_i\in M_i} \operatorname{int} T_i^{-1}(y_i)$,*
- (iii) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
- (iv) *for any continuous mapping $f_i : X \to Y_i$, the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon \varphi_i(x, f_i(x), z_i) \leqslant 0\}$,*
- (v) *for each $(x, y_i) \in X \times Y_i$, $z_i \mapsto \varphi_i(x, y_i, z_i)$ is G-quasi-convex,*
- (vi) *for any continuous mapping $f_i : X \to Y_i$ and $x \in X$, $\varphi_i(x, f_i(x), x_i) > 0$.*

Then there exist $\hat{x} \in X$ *and* $\hat{y} \in Y = \prod_{i \in I} Y_i$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) > 0 \quad \forall z_i \in A_i(\hat{x}).$$

Proof. For each $i \in I$, let $Z_i = [-\infty, +\infty]$ and $C_i(x) = [-\infty, 0]$ for each $x \in X$. Noting that φ_i is a single-valued function, it is easy to check that all conditions of Theorem 2.4 are satisfied. By Theorem 2.4, there exist $\hat{x} \in X$ and $\hat{y} \in Y = \prod_{i \in I} Y_i$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) > 0 \quad \forall z_i \in A_i(\hat{x}). \qquad \square$$

Theorem 2.5. *Let* $(X_i, D_i; \Gamma_i)$ *be a family of locally G-convex uniform spaces,* $(Y_i, D_i'; \Gamma_i')_{i \in I}$ *be a family of G-convex spaces and* $\{Z_i\}_{i \in I}$ *be a family of topological spaces. For each* $i \in I$, *let* $A_i : X \to 2^{X_i}$, $T_i : X \to 2^{Y_i}$, $\varphi_i : X \times Y_i \times X_i \to 2^{Z_i}$, *and* $C_i : X \to 2^{Z_i}$ *be set-valued mappings such that for each* $i \in I$,

- (i) T_i *has nonempty* Γ_i'*-convex values,*
- (ii) *there exists* $M_i = \{y_{i,0}, \ldots, y_{i,n_i}\} \in \mathcal{F}(D_i')$ *such that* $X = \bigcup_{y_i \in M_i} \operatorname{int} T_i^{-1}(y_i)$,
- (iii) A_i *is an u.s.c. compact mapping with nonempty closed* Γ_i*-convex values,*
- (iv) *for any continuous mapping* $f_i : X \to Y_i$, *the set* $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ *is open in* X *where the mapping* $P_i : X \to 2^{X_i}$ *is defined by* $P_i(x) = \{z_i \in X_i\colon \varphi_i(x, f_i(x), z_i) \nsubseteq C_i(x)\}$,
- (v) $\varphi_i(x, y_i, z_i)$ *is an u.s.c. compact mapping with closed values,*
- (vi) *the mapping* C_i *has open graph,*
- (vii) *for each* $(x, y_i) \in X \times Y_i$, $z_i \mapsto \varphi_i(x, y_i, z_i)$ *is G-quasi-concave with respect to* C_i,
- (viii) *for any continuous mapping* $f_i : X \to Y_i$ *and* $x \in X$, $\varphi_i(x, f_i(x), x_i) \subseteq C_i(x)$.

Then there exist $\hat{x} \in X$ *and* $\hat{y} \in Y = \prod_{i \in I} Y_i$ *such that for each* $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

Proof. By conditions (i), (ii) and Lemma 2.1 for each $i \in I$, T_i has a continuous selection $f_i : X \to Y_i$. For each $i \in I$, define a set-valued mapping $F_i : X \times X_i \to 2^{Z_i}$ by

$$F_i(x, z_i) = \varphi_i\big(x_i f_i(x), z_i\big) \quad \forall (x, z_i) \in X \times X_i.$$

By conditions (iii)–(viii), it is easy to see that all conditions of Theorem 2.3 are satisfied. By Theorem 2.3, there exists $\hat{x} \in X$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}) \quad \text{and} \quad F_i(\hat{x}, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}).$$

For each $i \in I$, let $\hat{y}_i = f_i(\hat{x})$. Then $\hat{y} \in Y = \prod_{i \in I} Y_i$. Noting that f_i is a continuous selection of T_i we conclude that for each $i \in I$, $\hat{y}_i = f_i(\hat{x}) \in T_i(\hat{x})$. It follows that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) \subseteq C_i(\hat{x}) \quad \forall z_i \in A_i(\hat{x}). \qquad \square$$

Corollary 2.4. *Let* $(X_i, D_i; \Gamma_i)$ *be a family of locally G-convex uniform spaces such that* $X = \prod_{i \in I} X_i$ *is a normal space. Let* $(Y_i, D_i', \Gamma_i')_{i \in I}$ *be a family of G-convex spaces. For each* $i \in I$, $A_i : X = \prod_{i \in I} X_i \to 2^{X_i}$ *and* $T : X \to 2^{Y_i}$. *Let* $\varphi : X \times Y_i \times X_i \to [-\infty, +\infty]$ *be a continuous function. Suppose that for each* $i \in I$,

(i) *T_i has nonempty Γ_i'-convex values,*
(ii) *there exists $M_i = \{y_{i,0}, \dots, y_{i,n_i}\} \in \mathcal{F}(D_i')$ such that $X = \bigcup_{y_i \in M_i} \operatorname{int} T_i^{-1}(y_i)$,*
(iii) *A_i is an u.s.c. compact mapping with nonempty closed Γ_i-convex values,*
(iv) *for any continuous mapping $f_i : X \to Y_i$, the set $E_i = \{x \in X\colon A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X where the mapping $P_i : X \to 2^{X_i}$ is defined by $P_i(x) = \{z_i \in X_i\colon \varphi_i(x, f_i(x), z_i) \geqslant 0\}$,*
(v) *for each $(x, y_i) \in X \times Y_i$, $z_i \mapsto \varphi_i(x, y_i, z_i)$ is G-quasi-concave,*
(vi) *for any continuous mapping $f_i : X \to Y_i$ and $x \in X$, $\varphi_i(x, f_i(x), x_i) < 0$.*

Then there exist $\hat{x} \in X$ and $\hat{y} \in Y = \prod_{i \in I} Y_i$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \textit{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) < 0 \quad \forall z_i \in A_i(\hat{x}).$$

Proof. For each $i \in I$, let $Z_i = [-\infty, +\infty]$ and $C_i(x) = [-\infty, 0)$ for each $x \in X$. Noting that φ is a single-valued function, it is easy to check that all conditions of Theorem 2.5 are satisfied. Hence by Theorem 2.5 there exist $\hat{x} \in X$ and $\hat{y} \in Y = \prod_{i \in I} Y_i$ such that for each $i \in I$,

$$\hat{x}_i \in A_i(\hat{x}), \quad \hat{y}_i \in T_i(\hat{x}) \quad \text{and} \quad \varphi_i(\hat{x}, \hat{y}_i, z_i) < 0 \quad \forall z_i \in A_i(\hat{x}).$$

The proof is now complete. □

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