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Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces ☆

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Abstract

In this paper, we introduce the concept of τ-function which generalizes the concept of w-distance studied in the literature. We establish a generalized Ekeland's variational principle in the setting of lower semicontinuous from above and τ-functions. As applications of our Ekeland's variational principle, we derive generalized Caristi's (common) fixed point theorems, a generalized Takahashi's nonconvex minimization theorem, a nonconvex minimax theorem, a nonconvex equilibrium theorem and a generalized flower petal theorem for lower semicontinuous from above functions or lower semicontinuous functions in the complete metric spaces. We also prove that these theorems also imply our Ekeland's variational principle.

Keywords: τ-Function; Generalized Ekeland's variational principle; Lower semicontinuous from above function; Generalized Caristi's (common) fixed point theorem; Nonconvex minimax theorem; Nonconvex equilibrium theorem; Generalized flower petal theorem

1. Introduction

In 1972, Ekeland [10] proved the celebrated variational principle for approximate solutions of nonconvex minimization problems. It is well known that the primitive Ekeland's varia-

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tional principle [10–12] (see also [8,31,32]) is equivalent to the Caristi's fixed point theorem [3–8,19,22,25,26], to the drop theorem [17,25], to the petal theorem [17,25], and to the Takahashi's nonconvex minimization theorem [1,15,18,27,31,32] and that by virtue of these equivalences it has found interesting applications in a significant way in various fields of applied mathematics. A number of generalizations of these results have been investigated by several authors; see [1,3–5,9,13–18,27–34] and references therein. McLinden [21] obtained some applications of Ekeland's variational principles to minimax problems in Banach spaces and Oettli and Théra [23] and Park [24] gave some equilibrium formulations of Ekeland's variational principles. Recently, Kada et al. [18], Amemiya and Takahashi [1], Shioji et al. [26], Suzuki [27–29] and Suzuki and Takahashi [30] improved and generalized the Takahashi's nonconvex minimization theorem, Caristi's fixed point theorem and Ekeland's variational principle by using w-distances or τ-distances. In 2002, Chen et al. [7] introduced the concept of lower semicontinuous from above functions and use them to improve the Ekeland's variational principle and Caristi's fixed point theorem.

In Section 2, we first introduce the concept of τ-function which generalizes the concept of w-distance studied by Kada et al. [18], then we establish a generalized Ekeland's variational principle for lower semicontinuous from above functions. We also derive generalized Caristi's (common) fixed point theorems. In Section 3, we establish a nonconvex maximum element theorem in complete metric spaces. We give generalized Takahashi's nonconvex minimization theorems and show the equivalence relations between them. Applying generalized Ekeland's variational principles, we establish nonconvex minimax theorems and nonconvex equilibrium theorems for lower semicontinuous from above functions in complete metric spaces. We also deduce other new equivalence formulations of generalized Ekeland's variational principles and our results include some known results of [1] and many results in the literature as special cases. Finally, we establish generalized flower petal theorems.

Consequently, our new results improve and generalize a lot of well-known works due to Kada et al. [18], Amemiya and Takahashi [1], Shioji et al. [26], Suzuki [27–29], Suzuki and Takahashi [30], Chen et al. [7], Zhong [34] and others, with different proofs and some new nonconvex existence theorems in complete metric spaces are also established.

2. Ekeland's variational principle

Throughout the paper, unless specified otherwise, (X,d) is a metric space and $\varphi:(-\infty,\infty]\to(0,\infty)$ is a nondecreasing function.

An extended real-valued function $f:X\to(-\infty,\infty]$ is said to be

(i) *lower semicontinuous from above* (in short lsca) *at* $x_0\in X$ [7] if for any sequence $\{x_n\}$ in X with $x_n\to x_0$ and $f(x_1)\geqslant f(x_2)\geqslant\cdots\geqslant f(x_n)\geqslant\cdots$ implies that $f(x_0)\leqslant\lim_{n\to\infty}f(x_n)$;
(ii) *upper semicontinuous from below* (in short uscb) *at* $x_0\in X$ if for any sequence $\{x_n\}$ in X with $x_n\to x_0$ and $f(x_1)\leqslant f(x_2)\leqslant\cdots\leqslant f(x_n)\leqslant\cdots$ implies that $f(x_0)\geqslant\lim_{n\to\infty}f(x_n)$.

The function f is said to be lsca (respectively uscb) on X if f is lsca (respectively uscb) at every point of X. The function f is said to be proper if $f\not\equiv\infty$.

The following definition of τ-function is different from the definition of τ-distance, it is a generalization of w-distance in [18].

Definition 2.1. A function $p : X \times X \to [0, \infty)$ is said to be a *τ-function* if the following conditions hold:

(τ1) for all $x, y, z \in X$, $p(x, z) \leqslant p(x, y) + p(y, z)$;
(τ2) if $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $p(x, y_n) \leqslant M$ for some $M = M(x) > 0$ then $p(x, y) \leqslant M$;
(τ3) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > n\} = 0$, and if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$, then $\lim_{n\to\infty} d(x_n, y_n) = 0$;
(τ4) for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

It is known [18,32] that if p is a w-distance on $X \times X$, then for $x, y, z \in X$, $p(x, y) = 0$ and $p(x, z) = 0$ imply $y = z$.

Remark 2.1. Every w-distance, introduced and studied by Kada et al. [18] (see also [26,30,32]), is a τ-function.

Indeed, let p be a w-distance on $X \times X$. Clearly, (τ1) and (τ4) hold. If $x \in X$ and $\{y_n\}$ in X with $\lim_{n\to\infty} y_n = y$ such that $p(x, y_n) \leqslant M$ for some $M = M(x) > 0$, then (by (w2) [18]), $p(x, y) \leqslant \underline{\lim}_{n\to\infty} p(x, y_n) \leqslant M$. Therefore ($\tau$2) holds. Let $\{x_n\}$ be a sequence in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > n\} = 0$ and there exists $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n, y_n) = 0$. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n, x_{n+1}) \leqslant \delta/2$ and $p(x_n, y_n) < \delta/2$ whenever $n \geqslant n_0$. So $p(x_n, y_{n+1}) \leqslant p(x_n, x_{n+1}) + p(x_{n+1}, y_{n+1}) < \delta$ whenever $n \geqslant n_0$. Then (by (w3) [18]), $d(x_{n+1}, y_{n+1}) < \varepsilon$ whenever $n \geqslant n_0$. Hence $\lim_{n\to\infty} d(x_n, y_n) = 0$ and (τ3) holds. Therefore, p is a τ-function on $X \times X$.

Lemma 2.1. *Let p be a τ-function on $X \times X$. If a sequence $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > m\} = 0$, then $\{x_n\}$ is a Cauchy sequence in X.*

Proof. Let $\{x_n\}$ in X with $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > n\} = 0$, then $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$. Let $y_n = x_{n+1}$, $n \in \mathbb{N}$, we have $\lim_{n\to\infty} p(x_n, y_n) = 0$. By ($\tau$3), we obtain $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. This shows that $\{x_n\}$ is a Cauchy sequence in X. $\square$

Lemma 2.2. *Let $f : X \to (-\infty, \infty]$ be a function and p be a τ-function on $X \times X$. For each $x \in X$, let*

$$S(x) = \big\{y \in X\colon\ y \neq x,\ p(x, y) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(y)\big)\big\}.$$

If $S(x)$ is nonempty for some $x \in X$, then for each $y \in S(x)$, we have $f(y) \leqslant f(x)$ and $S(y) \subseteq S(x)$.

Proof. Let $y \in S(x)$. Then $y \neq x$ and $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. Since $p(x, y) \geqslant 0$ for any $x, y \in X$ and φ is nondecreasing with values in $(0, \infty)$, we have $f(x) \geqslant f(y)$. If $S(y) = \emptyset$, then we are done. If $S(y) \neq \emptyset$, let $z \in S(y)$. Then $z \neq y$ and $p(y, z) \leqslant \varphi(f(y))(f(y) - f(z))$. By the same arguments as above, we have $f(y) \geqslant f(z)$. Therefore, $f(x) \geqslant f(y) \geqslant f(z)$ and

$$p(x, z) \leqslant p(x, y) + p(y, z) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(z)\big).$$

Also we have $z \neq x$. Indeed, if $z = x$ then $p(x, z) = 0$. Since

$$p(x, y) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(y)\big) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(z)\big) = 0,$$

we have $p(x, y) = 0$. By $(\tau 4)$, we have $y = z$, which is a contradiction. Therefore $z \in S(x)$ and hence $S(y) \subseteq S(x)$. $\square$

Now we establish an intersection result which plays a key role in the proof of the main result of this paper.

Proposition 2.1. *Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded below function and p be a τ-function on $X \times X$. For each $x \in X$, let $S(x)$ be the same as in Lemma 2.2. If $\{x_n\}$ is a sequence in X such that $S(x_n)$ is nonempty and $x_{n+1} \in S(x_n)$ for all $n \in \mathbb{N}$, then there exists $x_0 \in X$ such that $x_n \to x_0$ and $x_0 \in \bigcap_{n=1}^{\infty} S(x_n)$.*

Moreover, if $f(x_{n+1}) \leqslant \inf_{z \in S(x_n)} f(z) + 1/n$ for all $n \in \mathbb{N}$, then $\bigcap_{n=1}^{\infty} S(x_n)$ contains precisely one point.

Proof. We first prove that $\{x_n\}$ is a Cauchy sequence. Since $x_{n+1} \in S(x_n)$, we have $f(x_n) \geqslant f(x_{n+1})$ for each $n \in \mathbb{N}$ and so $\{f(x_n)\}$ is nonincreasing. Also since f is bounded below, $\lim_{n\to\infty} f(x_n)$ exists. Let $r = \lim_{n\to\infty} f(x_n) = \inf_{n\in\mathbb{N}} f(x_n)$, then $f(x_n) \geqslant r$ for all $n \in \mathbb{N}$. We claim that $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > n\} = 0$. Since φ is nondecreasing, we have

$$p(x_n, x_m) \leqslant \sum_{j=n}^{m-1} p(x_j, x_{j+1}) \leqslant \varphi\big(f(x_1)\big)\big(f(x_n) - r\big), \quad \text{for } m, n \in \mathbb{N} \text{ with } m > n.$$

Let $\alpha_n = \varphi(f(x_1))(f(x_n) - r)$ for all $n \in \mathbb{N}$, then $\sup\{p(x_n, x_m)\colon m > n\} \leqslant \alpha_n$ for each $n \in \mathbb{N}$. Since $\lim_{n\to\infty} f(x_n) = r$, we have $\lim_{n\to\infty} \alpha_n = 0$ and $\lim_{n\to\infty} \sup\{p(x_n, x_m)\colon m > n\} = 0$. By Lemma 2.1, $\{x_n\}$ is a Cauchy sequence.

By the completeness of X, there exists $x_0 \in X$ such that $x_n \to x_0$. We claim that $x_0 \in \bigcap_{n=1}^{\infty} S(x_n)$.

Since f is lsca, it follows that $f(x_0) \leqslant \lim_{n\to\infty} f(x_n) = r \leqslant f(x_k)$ for all $k \in \mathbb{N}$. Let $n \in \mathbb{N}$ be fixed and for all $m \in \mathbb{N}$ with $m > n$, we have

$$p(x_n, x_m) \leqslant \sum_{j=n}^{m-1} p(x_j, x_{j+1}) \leqslant \varphi\big(f(x_n)\big)\big(f(x_n) - f(x_0)\big).$$

From $(\tau 2)$, we have

$$p(x_n, x_0) \leqslant \varphi\big(f(x_n)\big)\big(f(x_n) - f(x_0)\big) \quad \text{for all } n \in \mathbb{N}. \tag{2.1}$$

Also we have $x_0 \neq x_n$ for all $n \in \mathbb{N}$. Indeed, suppose that there exists $j \in \mathbb{N}$ such that $x_0 = x_j$. Since

$$p(x_j, x_{j+1}) \leqslant \varphi\big(f(x_j)\big)\big(f(x_j) - f(x_{j+1})\big) \leqslant \varphi\big(f(x_j)\big)\big(f(x_j) - f(x_0)\big) = 0,$$

we have $p(x_j, x_{j+1}) = 0$. Similarly, we obtain $p(x_{j+1}, x_{j+2}) = 0$. It follows that $p(x_j, x_{j+2}) = 0$. By $(\tau 4)$, we have $x_{j+1} = x_{j+2}$, which contradicts to the fact that $x_{j+1} \neq x_{j+2}$. This shows that $x_0 \neq x_n$ for all $n \in \mathbb{N}$. By (2.1), we have $x_0 \in \bigcap_{n=1}^{\infty} S(x_n)$ and hence $\bigcap_{n=1}^{\infty} S(x_n) \neq \emptyset$.

Moreover, if $f(x_{n+1}) \leqslant \inf_{z \in S(x_n)} f(z) + 1/n$ for all $n \in \mathbb{N}$, we want to show $\bigcap_{n=1}^{\infty} S(x_n) = \{x_0\}$. For each $w \in \bigcap_{n=1}^{\infty} S(x_n)$, we have

$$\begin{aligned} p(x_n, w) &\leqslant \varphi\big(f(x_n)\big)\big(f(x_n) - f(w)\big) \\ &\leqslant \varphi\big(f(x_1)\big)\Big(f(x_n) - \inf_{z \in S(x_n)} f(z)\Big) \\ &\leqslant \varphi\big(f(x_1)\big)\bigg(f(x_n) - f(x_{n+1}) + \frac{1}{n}\bigg) \end{aligned}$$

for all $n \in \mathbb{N}$. Let $\beta_n = \varphi(f(x_1))(f(x_n) - f(x_{n+1}) + 1/n)$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \beta_n = 0$ and hence $\lim_{n\to\infty} p(x_n, w) = 0$. By ($\tau$3), we obtain $x_n \to w$. By uniqueness, we have $w = x_0$. Hence $\bigcap_{n=1}^{\infty} S(x_n) = \{x_0\}$. $\square$

By applying Proposition 2.1, we obtain the following generalization of Ekeland's variational principle for lower semicontinuous from above functions.

Theorem 2.1 *(Generalized Ekeland's variational principle). Let $f : X \to (-\infty, \infty]$ be a proper lsca and bounded below function and p be a τ-function on $X \times X$. Then there exists $v \in X$ such that $p(v, x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$.*

Proof. On the contrary, assume that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. For each $x \in X$, let $S(x)$ be the same as in Lemma 2.2. Then $S(x) \neq \emptyset$ for each $x \in X$. Since f is proper, there exists $u \in X$ with $f(u) < \infty$. We define inductively a sequence $\{u_n\}$ in X, starting with $u_1 = u$. Then choose $u_2 \in S(u_1)$ such that $f(u_2) \leqslant \inf_{x\in S(u_1)} f(x) + 1$. Suppose that $u_n \in X$ is known, then choose $u_{n+1} \in S(u_n)$ such that $f(u_{n+1}) \leqslant \inf_{x\in S(u_n)} f(x) + 1/n$. From Proposition 2.1, there exists $x_0 \in X$ such that $\bigcap_{n=1}^{\infty} S(u_n) = \{x_0\}$. By Lemma 2.2, $S(x_0) \subseteq \bigcap_{n=1}^{\infty} S(u_n) = \{x_0\}$ and hence $S(x_0) = \{x_0\}$, which is a contradiction. Therefore, there exists $v \in X$ such that $p(v, x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. $\square$

As a first application of generalized Ekeland's variational principle, we derive the following generalized Caristi's (common) fixed point theorems for a family of multivalued maps.

Theorem 2.2 *(Generalized Caristi's common fixed point theorem for a family of multivalued maps). Let p and f be the same as in Theorem* 2.1. *Let I be any index set and for each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map with nonempty values such that for each $x \in X$, there exists $y = y(x, i) \in T_i(x)$ with*

$$p(x, y) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(y)\big). \tag{2.2}$$

Then there exists $v \in X$ such that $v \in \bigcap_{i\in I} T_i(v)$, that is, the family of multivalued maps $\{T_i\}_{i\in I}$ has a common fixed point in X, and $p(v, v) = 0$.

Proof. From Theorem 2.1, there exists $v \in X$ such that $p(v, x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. We claim that $v \in \bigcap_{i\in I} T_i(v)$ and $p(v, v) = 0$.

By the hypothesis, for each $i \in I$, there exists $w(v, i) \in T_i(v)$ such that $p(v, w(v, i)) \leqslant \varphi(f(v))(f(v) - f(w(v, i)))$. Then $w(v, i) = v$ for each $i \in I$. Indeed, if $w(v, i_0) \neq v$ for some $i_0 \in I$, then $p(v, w(v, i_0)) \leqslant \varphi(f(v))(f(v) - f(w(v, i_0))) < p(v, w(v, i_0))$, which leads to a contradiction. Hence $v = w(v, i) \in T_i(v)$ for all $i \in I$. Since $p(v, v) \leqslant \varphi(f(v))(f(v) - f(v)) = 0$, we obtain $p(v, v) = 0$. $\square$

If for each $i \in I$, T_i is a single-valued map, then the following result can be easily derived from the above theorem.

Corollary 2.1 *(Generalized Caristi's common fixed point theorem for a family of single-valued maps). Let p and f be the same as in Theorem* 2.1. *Let I be any index set and for each $i \in I$, let $g_i : X \to X$ be a single-valued map satisfying*

$$p\big(x, g_i(x)\big) \leqslant \varphi\big(f(x)\big)\big(f(x) - f\big(g_i(x)\big)\big), \tag{2.3}$$

for each $x \in X$. Then there exists $v \in X$ such that $g_i(v) = v$ for each $i \in I$ and $p(v, v) = 0$.

Remark 2.2.

(a) Corollary 2.1 implies Theorem 2.2.
Indeed, under the hypothesis of Theorem 2.2, for each $x \in X$, there exists $y(x, i) \in T_i(x)$ such that $p(x, y(x, i)) \leqslant \varphi(f(x))(f(x) - f(y(x, i)))$. For each $i \in I$, we set $g_i(x) = y(x, i)$. Then g_i is a single-valued map from X into itself satisfying $p(x, g_i(x)) \leqslant \varphi(f(x))(f(x) - f(g_i(x)))$ for all $x \in X$. By Corollary 2.1, there exists $v \in X$ such that $v = g_i(v) \in T_i(v)$ for each $i \in I$ and $p(v, v) = 0$.

(b) Theorem 2.2 implies Theorem 2.1.
Indeed, suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. Then for each $x \in X$, we can define a multivalued mapping $T : X \to 2^X \setminus \{\emptyset\}$ by

$$T(x) = \big\{y \in X:\ y \neq x,\ p(x, y) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(y)\big)\big\}.$$

By Theorem 2.2, T has a fixed point $v \in X$, that is, $v \in T(v)$. But $v \notin T(v)$ a contradiction.

In the rest of the paper, unless specified otherwise, let (X, d), p, f, and φ be the same as in Theorem 2.1 and let I be any index set.

Theorem 2.3 *(Nonconvex maximal element theorem for a family of multivalued maps). For each $i \in I$, let $T_i : X \to 2^X$ be a multivalued map. Assume that for each $(x, i) \in X \times I$ with $T_i(x) \neq \emptyset$, there exists $y = y(x, i) \in X$ with $y \neq x$ such that* (2.2) *holds. Then there exists $v \in X$ such that $T_i(v) = \emptyset$ for each $i \in I$.*

Proof. From Theorem 2.1, there exists $v \in X$ such that $p(v, x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. We claim that $T_i(v) = \emptyset$ for each $i \in I$. Suppose to the contrary that there exists $i_0 \in I$ such that $T_{i_0}(v) \neq \emptyset$. By hypothesis, there exists $w = w(v, i_0) \in X$ with $w \neq v$ such that $p(v, w) \leqslant \varphi(f(v))(f(v) - f(w))$. It follows that

$$p(v, w) \leqslant \varphi\big(f(v)\big)\big(f(v) - f(w)\big) < p(v, w),$$

which leads to a contradiction. Hence $T_i(v) = \emptyset$ for each $i \in I$. □

Remark 2.3. Theorem 2.3 implies Theorem 2.1.

Indeed, suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. For each $x \in X$, define a multivalued map by

$$T(x) = \big\{y \in X:\ y \neq x,\ p(x, y) \leqslant \varphi\big(f(x)\big)\big(f(x) - f(y)\big)\big\}.$$

Then $T(x) \neq \emptyset$ for all $x \in X$. But from Theorem 2.3, there exists $v \in X$ such that $T(v) = \emptyset$, a contradiction.

3. Nonconvex optimization and minimax theorems

Theorem 3.1 *(Generalized Takahashi's nonconvex minimization theorem). Suppose that for any $x \in X$ with $f(x) > \inf_{z\in X} f(z)$ there exists $y \in X$ with $y \neq x$ such that* (2.2) *holds. Then there exists $v \in X$ such that $f(v) = \inf_{z\in X} f(z)$.*

Proof. From Theorem 2.1, there exists $v \in X$ such that

$$p(v,x) > \varphi\big(f(v)\big)\big(f(v) - f(x)\big) \quad \text{for all } x \in X \text{ with } x \neq v.$$

We claim that $f(v) = \inf_{a\in X} f(a)$.

Suppose to the contrary that $f(v) > \inf_{z\in X} f(z)$. By our assumption, there exists $y = y(v) \in X$ with $y \neq v$ such that

$$p(v,y) \leqslant \varphi\big(f(v)\big)\big(f(v) - f(y)\big).$$

Then we have

$$p(v,y) \leqslant \varphi\big(f(v)\big)\big(f(v) - f(y)\big) < p(v,y)$$

which leads to a contradiction. □

Remark 3.1. Theorem 3.1 implies Theorem 2.1.

Indeed, suppose that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x,y) \leqslant \varphi(f(x))(f(x) - f(y))$. Then, by Theorem 3.1, there exists $v \in X$ such that $f(v) = \inf_{x\in X} f(x)$. By our supposition, there exists $w \in X$ with $w \neq v$ such that $p(v,w) \leqslant \varphi(f(v))(f(v) - f(w)) \leqslant 0$. Hence $p(v,w) = 0$ and $f(v) = f(w) = \inf_{x\in X} f(x)$. There exists $z \in X$ with $z \neq w$ such that $p(w,z) \leqslant \varphi(f(w))(f(w) - f(z)) \leqslant 0$. So we also have $p(w,z) = 0$ and $f(v) = f(w) = f(z) = \inf_{x\in X} f(x)$. Since $p(v,z) \leqslant p(v,w) + p(w,z) = 0$, $p(v,z) = 0$. By $(\tau 4)$, we have $w = z$, which leads to a contradiction.

Remark 3.2. [18, Theorem 1] and [27, Theorem 5] are special cases of Theorem 3.1.

Theorem 3.2 *(Nonconvex minimax theorem). Let $F : X \times X \to (-\infty, \infty]$ be a function such that it is proper and lsca and bounded below in the first argument. Suppose that for each $x \in X$ with $\{u \in X\colon F(x,u) > \inf_{a\in X} F(a,u)\} \neq \emptyset$, there exists $y = y(x) \in X$ with $y \neq x$ such that*

$$\begin{aligned} &p(x,y) \leqslant \varphi\big(F(x,w)\big)\big(F(x,w) - F(y,w)\big) \\ &\quad \textit{for all } w \in \Big\{u \in X\colon F(x,u) > \inf_{a\in X} F(a,u)\Big\}. \end{aligned} \tag{3.1}$$

Then $\inf_{x\in X} \sup_{y\in X} F(x,y) = \sup_{y\in X} \inf_{x\in X} F(x,y)$.

Proof. From the assumption of Theorem 3.2 follows from Theorem 3.1,

$$\forall y \in X,\ \exists x(y) \in X\colon \quad F\big(x(y), y\big) = \inf_{x\in X} F(x,y).$$

Taking the supremum over y on both sides yields

$$\sup_{y\in X} F\big(x(y), y\big) = \sup_{y\in X} \inf_{x\in X} F(x,y).$$

Replacing $x(y)$ by an arbitrary $x \in X$ and taking the inf, we obtain

$$\inf_{x\in X}\sup_{y\in X} F(x,y)=\sup_{y\in X}\inf_{x\in X} F(x,y).$$

This completes the proof. □

Remark 3.3. The convexity assumptions on the sets or on the functions are essential in many existing general topological minimax theorems. McLinden [21] obtained some applications of Ekeland's variational principles to minimax problems in Banach spaces. The results in [21] are patterned after Rockfellar's augmented version of Ekeland's variational principle, in which additional informations of subgradient type are extracted from the basic Ekeland's inequality. Note that the assumption and conclusion of Theorem 3.2 is different from [21]. Ansari et al. [2] and Lin [20] studied minimax theorems for a family of multivalued mappings in locally convex topological vector spaces. Certain convexity assumptions are assumed in [2,20] and references therein.

The following result is a nonconvex equilibrium theorem in complete metric spaces.

Theorem 3.3 *(Nonconvex equilibrium theorem). Let F and φ be the same as in Theorem* 3.2. *Suppose that for each $x\in X$, with $\{u\in X\colon\ F(x,u)<0\}\neq\emptyset$, there exists $y=y(x)\in X$ with $y\neq x$ such that* (3.1) *holds for all $w\in X$. Then there exists $v\in X$ such that$F(v,y)\geqslant 0$ for all $y\in X$.*

Proof. Applying Theorem 2.1, for each $z\in X$, there exists $v(z)\in X$ such that $p(v(z),x)>\varphi(F(v(z),z))(F(v(z),z)-F(x,z))$ for all $x\in X$ with $x\neq v(z)$. We claim that there exists $v\in X$ such that $F(v,y)\geqslant 0$ for all $y\in X$. Suppose to the contrary for each $x\in X$ there exists $y\in X$ such that $F(x,y)<0$. Then for each $x\in X$ the set $\{u\in X\colon\ F(x,u)<0\}\neq\emptyset$. By assumption, there exists $y=y(v(z))$, $y\neq v(z)$ such that $p(v(z),y)\leqslant\varphi(F(v(z),z))(F(v(z),z)-F(y,z))$. This leads to a contradiction. Therefore, there exists $x\in X$ such that $F(v,y)\geqslant 0$ for all $y\in X$. □

Example 3.1. Let $X=[0,1]$ with the metric $d(x,y)=|x-y|$. Then (X,d) is a complete metric space. Let a and b be positive real numbers with $a\geqslant b$. Let $F: X\times X\to\mathbb{R}$ be defined by $F(x,y)=ax-by$. It is easy to see that for each $y\in X$, the function $x\to F(x,y)$ is a proper lsc and bounded below function on X and $F(1,y)\geqslant 0$ for all $y\in X$. In fact, $F(x,y)\geqslant 0$ for all $x\in[\frac{b}{a},1]$ and all $y\in X$. Note that for each $x\in[0,\frac{b}{a})$, $F(x,y)=ax-by<0$ for all $y\in[\frac{a}{b}x,1]$. Hence $\{u\in X\colon\ F(x,u)<0\}\neq\emptyset$ for all $x\in[0,\frac{b}{a})$. For any $x\geqslant y$, $x,y\in X$, we have $x-y=\frac{1}{a}\{(ax-bu)-(ay-bu)\}$ for all $u\in X$. Define a nondecreasing function $\varphi:[0,+\infty)\to[0,+\infty)$ by $\varphi(t)=\frac{1}{a}$. Hence $d(x,y)\leqslant\varphi(F(x,u))(F(x,u)-F(y,u))$ for all $x\geqslant y$, $x,y,u\in X$. By Theorem 3.3, we also show that there exists $v\in X$ such that $F(v,y)\geqslant 0$ for all $y\in X$.

Remark 3.4. Oettli and Théra [23] and Park [24] gave some equilibrium formulations of Ekeland's variational principles. But note that, in [24], the author assumed that

(a) $F(x,z)\leqslant F(x,y)+F(y,z)$ for all $x,y,z\in X$;
(b) for any $x\in X$, $F(x,\cdot): X\to(-\infty,\infty]$ is lsc;
(c) there exists $x_0\in X$ such that $\inf_{y\in X}F(x_0,y)>-\infty$

and, in [23], the authors assumed that $F(x,x)=0$ for any $x\in X$ in addition to conditions (a)–(b). So Theorem 3.3 is different from their one.

Theorem 3.4. *For each $i \in I$, let $T_i : X \multimap X$ be multivalued maps with nonempty values, $g_i, h_i : X \times X \to \mathbb{R}$ be functions and $\{a_i\}$ and $\{b_i\}$ be families of real numbers. Assume that the following conditions hold*:

(i) *for each $(x, i) \in X \times I$, there exists $y = y(x, i) \in T_i(x)$ such that $g_i(x, y) \geqslant a_i$ and $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$;*
(ii) *for each $(u, i) \in X \times I$, there exists $w = w(u, i) \in T_i(u)$ such that $h_i(u, w) \leqslant b_i$ and $p(u, w) \leqslant \varphi(f(u))(f(u) - f(w))$.*

Then there exists $x_0 \in T_i(x_0)$ such that $g_i(x_0, x_0) \geqslant a_i$ and $h_i(x_0, x_0) \leqslant b_i$ for all $i \in I$ and $p(x_0, x_0) = 0$.

Proof. Applying Theorem 2.1, there exists $v \in X$ such that $p(v, x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. For each $i \in I$, (i) implies that there exists $w_1 = w_1(v, i) \in T_i(v)$ such that $g_i(v, w_1) \geqslant a_i$ and $p(v, w_1) \leqslant \varphi(f(v))(f(v) - f(w_1))$ and (ii) implies that there exists $w_2 = w_2(v, i) \in T_i(v)$ such that $h_i(v, w_2) \leqslant b_i$ and $p(v, w_2) \leqslant \varphi(f(v))(f(v) - f(w_2))$. If $w_1 \neq v$, then $p(v, w_1) \leqslant \varphi(f(v))(f(v) - f(w_1)) < p(v, w_1)$, which leads to a contradiction. Hence $w_1 = v$. Similarly, we have $w_2 = v$. Since $p(v, v) \leqslant \varphi(f(v))(f(v) - f(v)) = 0$, we obtain $p(v, v) = 0$. □

Remark 3.5.

(a) In Theorem 3.4, if $g_i = h_i = F_i$ and $a_i = b_i = c_i$, then there exists $x_0 \in T_i(x_0)$ such that $F_i(x_0, x_0) = c_i$ for all $i \in I$ and $p(x_0, x_0) = 0$.
(b) In (a), if $T_i(x) = X$ for all $x \in X$, then there exists $x_0 \in X$ such that $F_i(x_0, x_0) = c_i$ for all $i \in I$ and $P_i(x_0, x_0) = 0$.
(c) [1, Theorem 3.1] is a special case of Theorem 3.4.

Remark 3.6. Theorem 3.4 implies Theorem 2.1.

Indeed, assume that for each $x \in X$, there exists $y \in X$ with $y \neq x$ such that $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. Define a multivalued map $T : X \multimap X \setminus \{\emptyset\}$ by $T(x) = \{y \in X\colon\ y \neq x\}$ and a function $F : X \times X \to \mathbb{R}$ by $F(x, y) = \chi_{T(x)}(y)$, where χ_A is the characteristic function for an arbitrary set A. Note that $y \in T(x) \Leftrightarrow F(x, y) = 1$. Thus for each $x \in X$, there exists $y \in X$ such that $F(x, y) = 1$ and $p(x, y) \leqslant \varphi(f(x))(f(x) - f(y))$. By Remark 3.5(a) with $c = 1$, there exists $x_0 \in X$ such that $F(x_0, x_0) = 1$ and $p(x_0, x_0) = 0$. Hence we have $x_0 \in T(x_0)$. This is a contradiction and the proof is completed.

4. Applications to flower petal theorems

In this section, we establish some applications to flower petal theorems.

Definition 4.1. Let (X, d) be a metric space and $a, b \in X$. Let $\kappa : X \to (0, \infty)$ be a function and p be a w-distance on X.

The (p, κ)-*flower petal* $P_\varepsilon(a, b)$ (in short $P_\varepsilon(a, b, \kappa)$) associated with $\varepsilon \in (0, \infty)$ and $a, b \in X$ is the set

$$P_\varepsilon(a, b, \kappa) = \bigl\{x \in X\colon\ \varepsilon p(a, x) \leqslant \kappa(a)\bigl(p(b, a) - p(b, x)\bigr)\bigr\}.$$

Obviously, if the w-distance p with $p(a, a) = 0$, then $P_\varepsilon(a, b, \kappa)$ is nonempty.

Lemma 4.1. *Let $\varepsilon > 0$ and p be a w-distance on X. Suppose that there exists $u \in X$ such that $f(u) < \infty$ and $p(u,u) = 0$. Then there exists $v \in X$ such that*

(i) $\varepsilon p(u,v) \leqslant \varphi(f(u))(f(u) - f(v))$;
(ii) $\varepsilon p(v,x) > \varphi(f(v))(f(v) - f(x))$ *for all* $x \in X$ *with* $x \neq v$.

Proof. Let $u \in X$ be given with $f(u) < +\infty$ and $p(u,u) = 0$. Put $Y = \{x \in X\colon\ \varepsilon p(u,x) \leqslant \varphi(f(u))(f(u) - f(x))\}$. Then (Y,d) is a nonempty complete metric space. By Theorem 2.1, there exists $v \in Y$ such that $\varepsilon p(v,x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in Y$ with $x \neq v$. For any $x \in X \setminus Y$, since $\varepsilon[p(u,v) + p(v,x)] \geqslant \varepsilon p(u,x) > \varphi(f(u))(f(u) - f(x)) \geqslant \varepsilon p(u,v) + \varphi(f(v))(f(v) - f(x))$, it follows that $\varepsilon p(v,x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X \setminus Y$. Hence $\varepsilon p(v,x) > \varphi(f(v))(f(v) - f(x))$ for all $x \in X$ with $x \neq v$. □

Theorem 4.1 *(Generalized flower petal theorem). Let M be a proper complete subset of a metric space (X,d) and $a \in M$. Let p be a w-distance on X with $p(a,a) = 0$. Suppose that $b \in X \setminus M$, $p(b,M) = \inf_{x \in M} p(b,x) \geqslant r$ and $p(b,a) = s > 0$ and there exists a function κ from X into $(0,\infty)$ satisfies $\kappa(x) = \varphi(p(b,x))$ for some nondecreasing function φ from $(-\infty,\infty]$ into $(0,\infty)$. Then for each $\varepsilon > 0$, there exists $v \in M \cap P_\varepsilon(a,b,\kappa)$ such that $P_\varepsilon(v,b,\kappa) \cap (M \setminus \{v\}) = \emptyset$. Moreover, $p(a,v) \leqslant \varepsilon^{-1}\kappa(a)(s - r)$.*

Proof. Let M be with the induced metric d. Hence (M,d) is a complete metric space. Define $f : M \to (-\infty,\infty]$ by $f(x) = p(b,x)$. Since $f(a) = p(b,a) = s < \infty$ and $r \leqslant p(b,M) = \inf_{x \in M} f(x)$, f is a proper lower semicontinuous and bounded below function. By Lemma 4.1, there exists $v \in M$ such that

(1) $\varepsilon p(a,v) \leqslant \kappa(a)(f(a) - f(v))$;
(2) $\varepsilon p(v,x) > \kappa(v)(f(v) - f(x))$ for all $x \in M$ with $x \neq v$.

By (1), we have $v \in M \cap P_\varepsilon(a,b,\kappa)$. Moreover, by (1) again, we also have $p(a,v) \leqslant \varepsilon^{-1}\kappa(a)(p(b,a) - p(b,v)) \leqslant \varepsilon^{-1}\kappa(a)(s - r)$. By (2), we obtain $\varepsilon p(v,x) > \kappa(v)(p(b,v) - p(b,x))$ for all $x \in M$ with $x \neq v$. Hence $x \notin P_\varepsilon(v,b,\kappa)$ for all $x \in M \setminus \{v\}$ or $P_\varepsilon(v,b,\kappa) \cap (M \setminus \{v\}) = \emptyset$. □

Remark 4.1. Under the assumptions of Theorem 4.1, we cannot verify $v \in P_\varepsilon(v,b,\kappa)_1$, but if we assume that the w-distance p with $p(x,x) = 0$ for all $x \in X$, then for each $\varepsilon > 0$, there exists $v \in M \cap P_\varepsilon(a,b,\kappa)$ such that $P_\varepsilon(v,b,\kappa) \cap M = \{v\}$.

In Theorem 4.1, if $\kappa(x) = 1$ for all $x \in X$, we have the primitive flower petal theorem in [25].

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