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ORIGINAL ARTICLE

Existences theorems of systems of vector quasi-equillibrium problems and mathematical programs with equilibrium constraint

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Abstract In this paper, we introduce systems of vector quasi-equilibrium problems and prove the existence of their solutions. As applications of our results, we derive the existence theorems for solution of system of vector quasi-saddle point problem, the existences theorems of a solution of system of generalized quasi-minimax inequalities, the mathematical program with equilibrium constraint, semi-infinite and bilevel problems.

Keywords Quasi-equilibrium problem · Mathematical programs with equilibrium constraint · Minimax theorem · Bilevel problem · Quasi-saddle point problem

1 Introduction

Let X be a convex subset of a real topological vector space E (in short t.v.s.) and $f: X \times X \to \mathbb{R}$ be a given function with $f(x,x) \geq 0$ for all $x \in X$. By equilibrium problem, Blum and Oettli [8] understood the problem of finding $\bar{x} \in X$ such that $f(\bar{x}, y) \geq 0$ for all $y \in X$. This problem contains optimization problems, Nash type equilibria problems, complementary problems, variational inequality problems and fixed point problems as special case.

In the recent past, systems of scalar (vector) equilibrium problems and scalar (vector) generalized equilibrium problems, systems of scalar (vector) quasi-equilibrium problems and scalar (vector) generalized quasi-equilibrium problems are used as tools to solve Nash equilibrium (for vector-valued functions) and Debreu type equilibrium problem (for vector-valued functions), system of optimization problems, system of mixed variational inequalities problems, system of saddle point problems and collective fixed point problems (see, e.g. [2–6, 10–12] and references therein.).

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The study of equilibrium problem is a new direction for the researchers (see, e.g. [2, 12, 13, 18] and references therein). There are many generalizations of this problems.

Let I be any index set. For each $i \in I$, let X_i and Y_i be nonempty convex subsets of locally convex topological vector space D_i and E_i, respectively, and Z_i, L be two real t.v.s.. Let $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$. Let $h : X \times Y \multimap L$ be a multivalued map. For each $i \in I$, let $C_i\colon\ X_i \multimap Z_i$ be a multivalued map such that $C_i(x_i)$ is a convex cone for all $x_i \in X_i$. For each $i \in I$, let $S_i\colon\ X \times Y \multimap X_i$, $T_i\colon\ X \multimap Y_i, f_i\colon\ X_i \times Y_i \times X_i \multimap Z_i$ and $g_i\colon\ X_i \times Y_i \times Y_i \multimap Z_i$ be multivalued maps. We consider the following problems of system of generalized vector quasi-equilibrium problems:

Find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, such that one of the following relations hold:

(1) $g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$.

(2) $g_i(\bar{x}_i, \bar{y}_i, v_i) \cap C_i(\bar{x}_i) \neq \emptyset$ for all $v_i \in T_i(\bar{x})$.

(3) $g_i(\bar{x}_i, \bar{y}_i, v_i) \cap -\mathrm{int}C_i(\bar{x}_i) = \emptyset$ for all $v_i \in T_i(\bar{x}_i)$.

(4) $g_i(\bar{x}_i, \bar{y}_i, v_i) \not\subseteq -\mathrm{int}C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$.

As application of our results, we establish of the following four types of system of simultaneous generalized vector quasi-equilibrium problems:

(5) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

(6) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \text{for all } v_i \in T_i(\bar{x}).$$

(7) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \cap -\mathrm{int}C_i(\bar{x}_i) = \emptyset \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap -\mathrm{int}C_i(\bar{x}_i) = \emptyset \quad \text{for all } v_i \in T_i(\bar{x}_i).$$

(8) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \not\subseteq -\mathrm{int}C_i(\bar{x}_i) \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \not\subseteq -\mathrm{int}C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

We derive several existence results for solutions of above mentioned problems and other similar problems. As applications in Sect. 4, we can obtain the systems of quasi-saddle point problem (in short, SVQSPP):

(SVQSPP): find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y}), \bar{y}_i \in T_i(\bar{x})$,

$$\varphi_i(x_i, \bar{y}_i) - \varphi_(\bar{x}_i, \bar{y}_i) \in C_i(\bar{x}_i) \quad \text{for all } x_i \in S_i(\bar{x}, \bar{y})$$

and

$$\varphi_i(\bar{x}_i, \bar{y}_i) - \varphi_i(\bar{x}_i, y_i) \in C_i(\bar{x}_i) \quad \text{for all} y_i \in T_i(\bar{x}).$$

For each $i \in I$, let f_i be a real-valued map. We also consider the following system of quasi-minimax problem:

Find $\bar{x} = (\bar{x}_i)_{i\in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$\min_{u_i\in S_i(\bar{x},\bar{y})} \max_{V_i\in T_i(\bar{x})} f_i(u_i, v_i) = f_i(\bar{x}_i, \bar{y}_i) = \max_{V_i\in T_i(\bar{x})} \min_{u_i\in S_i(\bar{x},\bar{y})} f_i(u_i, v_i).$$

Our approach are different from Lin [10] and Ansari et al. [2]. Moreover, in Sect. 5, we use the existence of the problems in Sect. 3 to study the mathematical programs with equilibrium constraint, semi-infinite and bilevel problems as following:

(MI) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \subseteq C_i(x_i), \text{ and } f_i(x_i, y_i, u_i) \subseteq C_i(x_i) \text{ for all } u_i \in S_i(x, y)\}$.

(MII) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), f_i(x_i, y_i) \subseteq C_i(x_i), \text{ and } g_i(x_i, y_i, v_i) \subseteq C_i(x_i) \text{ for all } v_i \in T_i(x)\}$.

(MIII) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \nsubseteq -\text{int}C_i(x_i), \text{ and } f_i(x_i, y_i, u_i) \nsubseteq -\text{int}C_i(x_i) \text{ for all } u_i \in S_i(x, y)\}$.

(MIV) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), f_i(x_i, y_i) \nsubseteq -\text{int}C_i(x_i), \text{ and } g_i(x_i, y_i, v_i) \nsubseteq -\text{int}C_i(x_i) \text{ for all } v_i \in T_i(x)\}$.

(MV) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \subseteq C_i(x_i), \text{ and } \varphi_i(x_i, u_i) \subseteq C_i(x_i) \text{for all } u_i \in S_i(x, y)\}$.

(MVI) Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \nsubseteq -\text{int}C_i(x_i), \text{ and } \varphi_i(x_i, u_i) \nsubseteq -\text{int}C_i(x_i) \text{ for all } u_i \in S_i(x, y)\}$.

If $Z_i = \mathbb{R}$, $C_i(x_i) = [0, \infty)$, h, f_i, g_i are single valued functions. (MI) and (MIII) will reduce to the following problem:

Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \geq 0, \text{ and } f_i(x_i, y_i, u_i) \geq 0 \text{ for all } u_i \in S_i(x, y)\}$.

(MII) and (MIV) will reduced to the following problem: Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I, x_i \in S_i(x, y), y_i \in T_i(x), f_i(x_i, y_i) \geq 0, \text{ and } g_i(x_i, y_i, v_i) \geq 0 \text{ for all } v_i \in T_i(x)\}$.

(MV) and (MVI) will reduce to the following problem:

Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), g_i(x_i, y_i) \geq 0, \text{ and } \varphi_i(x_i, u_i) \geq 0 \text{ for all } u_i \in S_i(x, y)\}$.

For the special case of our results, we also study the bilevel problem:

Min $h(K)$, where $K = \{(x, y) \in X \times Y \mid \text{for each } i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x), f_i(x_i, y_i) \geq 0,\ \varphi_i(x_i, y_i) \leq \varphi_i(x_i, v_i) \text{ for all } v_i \in T_i(x)\}$.

Recently Luo et al. [16], Lin and Still [14] and Lin [12] and references therein studied the mathematical program with equilibrium constraint problem, semi-infinite problem and bilevel problem, but our results and approach are different from [12, 14, 16]. In this paper, we study the existence theorems of various types of mathematical program with equilibrium constraint and semi-infinite problems with system of generalized vector quasi-equilibrium constraint.

2 Preliminaries

Let X and Y be nonempty subsets of a topological space E. We denote by 2^X the family of all subsets of the set X. A multivalued map $F: X \multimap Y$ is a function from X into 2^Y. Let X and Y be topological spaces and $T: X \multimap Y$ be a multivalued map. We call that T is *upper semicontinuous* (in short u.s.c.) (resp. *lower semicontinuous*, in short l.s.c.) at $x \in X$ if for every open set V containing $T(x)$ (resp. $T(x) \bigcap V \neq \emptyset$), there is an open set U containing x such that $T(u) \subseteq V$ (resp. $T(u) \cap V \neq \emptyset$) for all $u \in U$; T is u.s.c.(resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is *continuous* at x if T is both u.s.c. and l.s.c. at x; T is *closed* if GrT=$\{(x,y) \in X \times Y \mid y \in T(x)\}$ is closed in $X \times Y$; T is *compact* if there exists a compact set K such that $T(X) \subseteq K$.

Throughout this paper, all topological spaces are assumed to be Hausdorff. The following definitions and theorems are need in this paper.

Definition 2.1 Let X and Y be convex subset of a topological vector space. Let $g: X \times Y \multimap Z$, $h: X \multimap Z$ and $C: X \multimap Z$ be multivalued maps. h is said to be *convex* (resp. concave) if for all $x_1, x_2 \in X, \lambda \in [0,1], g(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda g(x_1) + (1-\lambda)g(x_2)$; (resp. $\lambda g(x_1) + (1-\lambda)g(x_2) \subseteq g(\lambda x_1 + (1-\lambda)x_2)$); g is said to be $C(x)-$*quasiconvex* if for any $x \in X, y_i, y_2 \in Y, \lambda \in [0,1]$, either

$$g(x,y_1) \subseteq g(x, \lambda y_1 + (1-\lambda)y_2) + C(x)$$

or

$$g(x,y_2) \subseteq g(x, \lambda y_1 + (1-\lambda)y_2) + C(x);$$

g is said to be $C(x)$-*quasiconcave-like* if for any $x \in X, y_1, y_2 \in Y$ and $\lambda \in [0,1]$, either

$$g(x, \lambda y_1 + (1-\lambda)y_2) \subseteq g(x,y_1) + C(x)$$

or

$$g(x, \lambda y_1 + (1-\lambda)y_2) \subseteq g(x,y_2) + C(x);$$

g is said to be $C(x)-$*quasiconcave* if for any $x \in X, y_1, y_2 \in Y$ and $\lambda \in [0,1]$, either

$$g(x,y_1) \subseteq g(x, \lambda y_1 + (1-\lambda)y_2) - C(x)$$

or

$$g(x,y_2) \subseteq g(x, \lambda y_1 + (1-\lambda)y_2) - C(x).$$

Definition 2.2 [9] Let X be a convex subset of a t.v.s. and Z be a t.v.s. Let $f: X \times X \multimap Z$ and $C: X \multimap Z$ be multivalued maps. Given any finite set $\Lambda = \{x_1, x_2, \ldots, x_n\}$ and any $x \in \text{co}\{x_1, x_2, \ldots, x_n\}$. f is said to be *strong type I C-diagonally quasiconvex* (SIC-DQC, in short) in the second argument if for some $x_i \in \Lambda, f(x,x_i) \subseteq C(x)$;$f$ is said to be *strong type II C-diagonally quasiconvex* (SIIC-DQC, in short) in the second argument if for some $x_i \in \Lambda, f(x, x_i) \cap C(x) \neq \emptyset$; f is said to be *weak type I C−diagonally quasiconvex* (WIC-DQC, in short) in the second argument if for some $x_i \in \Lambda$, $f(x, x_i) \cap (-\text{int}C(x)) = \emptyset$; f is said to be *weak type II C-diagonally quasiconvex* (WIIC-DQC, in short) in the second argument if for some $x_i \in \Lambda, f(x, x_i) \nsubseteq -\text{int}C(x)$.

Theorem 2.1 [1] *Let X and Y be Hausdorff topological spaces and $T : X \multimap Y$ be a multivalued map.*

(1) *If T is an u.s.c. multivalued map with closed values, then T is closed.*
(2) *If T is closed and Y is compact, then T is an u.s.c. multivalued map.*
(3) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

Theorem 2.2 [17] *Let T be a multivalued map of a topological space X into a topological space Y. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and for any net $\{x_\alpha\}$ in X converges to x, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ for every α and y_α converges to y.*

Theorem 2.3 [7] *Let K be a convex space, Z be a t.v.s., $F : K \times K \multimap Z$ and $C : K \multimap Z$ be multivalued maps such that $C(x)$ is a convex cone. Then F is $C(x)$-quaisconvex if and only if for any $x \in K$, $y_i \in K$, $t_i > 0$, $i = 1, 2, \dots, n$, $\sum_{i=1}^n t_i = 1$, then there exists $1 \leq j \leq n$ such that $F(x, y_i) \subseteq F\left(x, \sum_{i=1}^n t_i y_i\right) + C(x)$.*

Theorem 2.4 [15] *Let A be a nonempty compact subset of real t.v.s. Z, D a closed convex cone in Z such that $D \neq Z$, then* $\text{Min}_D A \neq \emptyset$.

3 The existence results for a solution of system of simultaneous generalized vector quasi-equilibrium problems

Let I be any index set. For each $i \in I$, let Z_i be a real t.v.s. and X_i, Y_i be nonempty closed convex subsets in locally convex t.v.s. D_i and E_i, respectively. Suppose that $C_i : X \multimap Z_i$ is a closed multivalued map and $C_i(x_i)$ is a nonempty convex cone for each $x_i \in X_i$ and $T_i : X \multimap Y_i$ is a compact continuous multivalued map with nonempty closed convex values. Throughout this paper we use these notations unless otherwise specified.

Theorem 3.1 *For each $i \in I$, suppose that*

(1) *$A_i : X \times Y \multimap X_i$ is a compact u.s.c. multivalued map with nonempty closed convex values*;
(2) *$g_i : X_i \times Y_i \times Y_i \multimap Z_i$ is a l.s.c multivalued map such that $g_i(x_i, y_i, y_i) \subseteq C_i(x_i)$ and for each $(x_i, v_i) \in X_i \times Y_i$, $y_i \multimap g_i(x_i, y_i, v_i)$ is $C_i(x_i)$-quasiconcave-like, and for each $(x_i, y_i) \in X_i \times Y_i$, $v_i \multimap g_i(x_i, y_i, v_i)$ is $C_i(x_i)$-quasiconvex.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

Proof For each $i \in I$, defined $H_i : X \multimap \overline{T_i(X)}$ by

$$H_i(x) = \{y_i \in T_i(x) \mid g_i(x_i, y_i, v_i) \subseteq C_i(x_i) \ \forall v_i \in T_i(x)\}.$$

For each $i \in I$ and $x \in X$, let $Q_i : T_i(x) \multimap T_i(x)$ be defined by $Q_i(v_i) = \{y_i \in T_i(x) \mid g_i(x_i, y_i, v_i) \subseteq C_i(x_i)\}$. Suppose there exists $i \in I$ and a finite set $\{v_i^1, v_i^2, \ldots, v_i^n\}$ in $T_i(x)$ such that $\mathrm{co}\{v_i^1, v_i^2, \ldots, v_i^n\} \nsubseteq \bigcup_{k=1}^n Q_i(v_i^k)$. Then there exists a $v_i^\lambda = \lambda_1 v_i^1 + \lambda_2 v_i^2 + \cdots + \lambda_n v_i^n \in \mathrm{co}\{v_i^1, v_i^2, \ldots, v_i^n\}$, where $\lambda_j \geq 0$ for $j = 1, 2, \ldots, n$ and $\sum_{j=1}^n \lambda_j = 1$, but $v_i^\lambda \notin \bigcup_{k=1}^n Q_i(v_i^k)$. Since $T_i(x)$ is convex, $v_i^\lambda \in \mathrm{co}\{v_i^1, v_i^2, \ldots, v_i^n\} \subseteq T_i(x)$. But $v_i^\lambda \notin Q_i(v_i^k)$ for all $k = 1, 2, \ldots, n$. So $g_i(x_i, v_i^\lambda, v_i^k) \nsubseteq C_i(x_i)$ for all $k = 1, 2, \ldots, n$. By (2) and Theorem 2.3, there exists $1 \leq j \leq n$ such that $g_i(x_i, v_i^\lambda, v_i^j) \subseteq g_i(x_i, v_i^\lambda, v_i^\lambda) + C_i(x_i) \subseteq C_i(x_i) + C_i(x_i) \subseteq C_i(x_i)$. This leads to a contradiction. So Q_i is a KKM map. For each $i \in I$, let $y_i \in \overline{Q_i(v_i)}$, then there exists a net $\{y_i^\alpha\}$ in $Q_i(v_i)$ such that $y_i^\alpha \to y_i$. So $y_i^\alpha \in T_i(x)$ and $g_i(x_i, y_i^\alpha, v_i) \subseteq C_i(x_i)$. Since $T_i(x)$ is closed, $y_i \in T_i(x)$. Let $z_i \in g_i(x_i, y_i, v_i)$. Since g_i is l.s.c., there exists a net $\{z_i^\alpha\}$ such that $z_i^\alpha \to z_i$ and $z_i^\alpha \in g_i(x_i, y_i^\alpha, v_i) \subseteq C_i(x_i)$. Since $C_i(x_i)$ is closed, $z_i \in C_i(x_i)$. So $g_i(x_i, y_i, v_i) \subseteq C_i(x_i)$, i.e., $y_i \in Q_i(v_i)$. Therefore, $Q_i(v_i)$ is closed. $Q_i(v_i)$ is closed in a compact set $\overline{T_i(X)}$, so $Q_i(v_i)$ is also compact. Then by KKM Theorem, $\bigcap_{v_i \in T_i(x)} Q_i(v_i) \neq \emptyset$, then we have $H_i(x) \neq \emptyset$. For each $i \in I$, let $y_i^1, y_i^2 \in H_i(x)$ and $\lambda \in [0, 1]$, then $y_i^1, y_i^2 \in T_i(x)$, $g_i(x_i, y_i^1, v_i) \subseteq C_i(x_i)$ for all $v_i \in T_i(x)$ and $g_i(x_i, y_i^2, v_i) \subseteq C_i(x_i)$ for all $v_i \in T_i(x)$. Let $y_i^\lambda = \lambda y_i^1 + (1-\lambda) y_i^2$. Since $T_i(x)$ is convex, $y_i^\lambda \in T_i(x)$. By (2), we have either $g_i(x_i, y_i^\lambda, v_i) \subseteq g_i(x_i, y_i^1, v_i) + C_i(x_i) \subseteq C_i(x_i) + C_i(x) \subseteq C_i(x_i)$ or $g_i(x_i, y_i^\lambda, v_i) \subseteq g_i(x_i, y_i^2, v_i) + C_i(x_i) \subseteq C_i(x_i) + C_i(x_i) \subseteq C_i(x_i)$. Therefore, $y_i^\lambda \in H_i(x)$, hence $H_i(x)$ is convex.

For each $i \in I$, let $(x, y_i) \in \overline{\mathrm{Gr} H_i}$, then there exists a net $\{(x^\alpha, y_i^\alpha)\}$ in $\mathrm{Gr} H_i$ such that $(x^\alpha, y_i^\alpha) \to (x, y_i)$. So we have $y_i^\alpha \in T_i(x^\alpha)$ and $g_i(x^\alpha, y^\alpha, v_i) \subseteq C_i(x^\alpha)$ for all $v_i \in T_i(x^\alpha)$. By Theorem 2.1 (1), $y_i \in T_i(x)$. Let $v_i \in T_i(x)$. Since T_i is l.s.c., there exists a net $\{v_i^\alpha\}$ such that $v_i^\alpha \to v_i$ and $v_i^\alpha \in T_i(x^\alpha)$. Let $z_i \in g_i(x_i, y_i, v_i)$. Since g_i is l.s.c., there exists a net $\{z_i^\alpha\}$ such that $z_i^\alpha \to z_i$ and $z_i^\alpha \in g_i(x_i^\alpha, y_i^\alpha, v_i^\alpha) \subseteq C_i(x_i^\alpha)$. Then $z_i \in C_i(x_i)$. Therefore, $g_i(x_i, y_i, v_i) \subseteq C_i(x_i)$ for all $v_i \in T_i(x)$, so we have $y_i \in H_i(x)$ and then H_i is closed. $H_i : X \multimap \overline{T_i(X)}$ is closed and $\overline{T_i(X)}$ is compact, so H_i is an u.s.c. multivalued map with nonempty closed convex values.

Now, defined $F : X \times Y \multimap X \times Y$ by $F(x, y) = \prod_{i \in I} \left[A_i(x, y) \times H_i(x)\right]$. Then F is an u.s.c. with nonempty closed convex values. By Himmelberg fixed point theorem, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$. It means that there exists $(\bar{x}, \bar{y}) \in X \times Y$such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

□

Corollary 3.1 *Suppose conditions (3)–(5) in Theorem* 3.1 *hold and for each* $i \in I$,

(i) $S_i : X \times Y \multimap X_i$ *is a compact continuous multivalued map with nonempty closed convex values;*
(ii) (a) $f_i : X_i \times Y_i \times X_i \multimap Z_i$ *is a l.s.c. multivalued map and convex in the first argument;*
  (b) *for each* $y_i \in Y_i$, *the function* $f_i(\cdot, y_i, \cdot)$ *is strong type I* C_i*-diagonally quasi-convex in the third argument and*
(iii) $C_i : X_i \multimap Z_i$ *is a concave closed multivalued map and* $C_i(x_i)$ *is a nonempty convex cone for each* $x_i \in X_i$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

Proof For each $i \in I$, define a multivalued map $M_i: X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x, y) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i, u_i) \subseteq C_i(w_i) \; \forall \, u_i \in S_i(x, y)\}.$$

Let G_i: $S_i(x, y) \multimap S_i(x, y)$ be defined by

$$G_i(u_i) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i, u_i) \subseteq C_i(w_i)\}.$$

With the same argument as in Theorem 3.1, we can show that G_i is a KKM map and $G_i(u_i)$ is closed. Since $G_i(u_i)$ is closed in the compact set $\overline{S_i(x, y)}$, $G_i(u_i)$ is also compact. Then by KKM Theorem, $\cap_{u_i \in S_i(x,y)} G_i(u_i) \neq \emptyset$, and hence $M_i(x, y) \neq \emptyset$. For each $i \in I$, let $w_i^1, w_i^2 \in M_i(x, y)$ and $\lambda \in [0, 1]$, then $w_i^1, w_i^2 \in S_i(x, y)$, $f_i(w_i^1, y_i, u_i) \subseteq C_i(w_i^1)$ and $f_i(w_i^2, y_i, u_i) \subseteq C_i(w_i^2)$ for all $u_i \in S_i(x, y)$. Let $w_i^\lambda = \lambda w_i^1 + (1 - \lambda) w_i^2$, then by (2a) and (3), $f_i(w_i^\lambda, y_i, u_i) \subseteq \lambda f_i(w_i^1, y_i, u_i) + (1 - \lambda) f_i(w_i^2, y_i, u_i) \subseteq \lambda C_i(w_i^1) + (1 - \lambda) C_i(w_i^2) \subseteq C_i(\lambda w_i^1 + (1 - \lambda) w_i^2) = C_i(w_i^\lambda)$. □

Since $S_i(x, y)$ is convex, $w_i^\lambda \in S_i(x, y)$. So $M_i(x, y)$ is convex.

Let $((x, y), w_i) \in \overline{\text{Gr}M_i}$, then there exist a net $\{((x^\alpha, y^\alpha), w_i^\alpha)\}$ in $\text{Gr}M_i$ such that $(x^\alpha, y^\alpha) \to (x, y)$ and $w_i^\alpha \to w_i$, so $w_i^\alpha \in S_i(x^\alpha, y^\alpha)$ and $f_i(w_i^\alpha, y_i^\alpha, u_i) \subseteq C_i(w_i^\alpha)$ for all $u_i \in S_i(x^\alpha, y^\alpha)$. By Theorem 2.1 (1), $w_i \in S_i(x, y)$. Let $u_i \in S_i(x, y)$. Since S_i is l.s.c., there exists a net $\{u_i^\alpha\}$ such that $u_i^\alpha \to u_i$ and $u_i^\alpha \in S_i(x^\alpha, y^\alpha)$. Let $z_i \in f_i(w_i, y_i, u_i)$. Since f_i is l.s.c., there exists a net $\{z_i^\alpha\}$ such that $z_i^\alpha \to z_i$ and $z_i^\alpha \in f_i(w_i^\alpha, y_i^\alpha, u_i^\alpha) \subseteq C_i(w_i^\alpha)$. Then we have $z_i \in C_i(w_i)$. So $f_i(w_i, y_i, u_i) \subseteq C_i(w_i)$ for all $u_i \in S_i(x, y)$. It means that $w_i \in M_i(x, y)$, so M_i is closed. By Theorem 2.1, M_i is a compact u.s.c. multivalued map with nonempty closed convex value. Then by Theorem 3.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in M_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and $g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$. i.e., there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i)$ for all $u_i \in S_i(\bar{x}, \bar{y})$ and $g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$.

The following corollary will have some applications to study the mathematical program with equilibrium constraint.

Corollary 3.2 *In Corollary* 3.1, *we replace* (2a) *and* (2b) *by*

(2a′) $S_i: X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values*;

(2b′) $f_i : X_i \times Y_i \multimap Z_i$ *is a l.s.c. multivalued map and convex in the first argument*; *for each* $x = (x_i)_{i \in I} \in X$ *and* $y = (y_i)_{i \in I} \in Y$, *there exists* $w_i \in S_i(x, y)$ *such that* $f_i(w_i, y_i) \subseteq C_i(w_i)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $f_i(\bar{x}_i, \bar{y}_i) \subseteq C_i(\bar{x}_i)$ and $g_i(\bar{x}_i, \bar{y}_i, v_i) \subseteq C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$.

Proof For each $i \in I$, defined $M_i: X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x, y) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i) \subseteq C_i(w_i)\}.$$

Following the similarly argument as in Corollary 3.1, we can show that M_i is a compact u.s.c. multivalued map with nonempty closed convex values. And then we obtain the result by Theorem 3.1. □

Theorem 3.2 *Suppose that conditions (i), (ii) and (iii) of Corollary 3.1. For each $i \in I$, suppose that $B_i : X \multimap Y_i$ is a compact u.s.c. multivalued map with nonempty closed convex values. Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in B_i(\bar{x})$,*

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i) \quad \textit{for all } u_i \in S_i(\bar{x}, \bar{y}).$$

Proof For each $i \in I$, define a multivalued map $M_i : X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x, y) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i, u_i) \subseteq C_i(w_i) \quad \forall\, u_i \in S_i(x, y)\}.$$

Then following the same argument as in Corollary 3.1, we have M_i is a compact u.s.c. multivalued map with nonempty closed convex values. Now, define $F : X \times Y \multimap X \times Y$ by $F(x, y) = \prod_{i \in I}[M_i(x, y) \times B_i(x)]$, then F is an u.s.c. multivalued map with nonempty closed convex values. By Himmelberg fixed point, we have the result.

Corollary 3.3 *For each $i \in I$, suppose that conditions (i) and (iii) of Corollary 3.1 and*

(1) (a) $f_i : X_i \times Y_i \times X_i \multimap Z_i$ is a l.s.c. multivalued map and convex in the first argument; (b) for each $y_i \in Y_i$, the function $f_i(\cdot, y_i, \cdot)$ is strong type I C_i-diagonally quasiconvex ;
(2) $g_i : X_i \times Y_i \multimap Z_i$ is a l.s.c. and for each $x_i \in X_i$, $y_i \multimap g_i(x_i, y_i)$ is $C_i(x_i)-$quasiconcave-like;
(3) for each $x = (x_i)_{i \in I} \in X$, there exists $y_i \in T_i(x)$ such that $g_i(x_i, y_i) \subseteq C_i(x_i)$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ such that $g_i(\bar{x}_i, \bar{y}_i) \subseteq C_i(\bar{x}_i)$ and $f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i)$ for all $u_i \in S_i(\bar{x}, \bar{y})$.

Proof For each $i \in I$, define $H_i : X \multimap \overline{T_i(X)}$ by

$$H_i(x) = \{y_i \in T_i(x) \mid g_i(x_i, y_i) \subseteq C_i(x_i)\}.$$

Then $H_i(x) \neq \emptyset$ by (iv). Let $(x, y_i) \in \overline{\mathrm{Gr} H_i}$, then there exists a net $\{(x^\alpha, y_i^\alpha)\}$ in $\mathrm{Gr} H_i$ such that $(x^\alpha, y_i^\alpha) \to (x, y_i)$. So we have $y_i^\alpha \in T_i(x^\alpha)$ and $g_i(x_i^\alpha, y_i^\alpha) \subseteq C_i(x_i^\alpha)$. By Theorem 2.1 (1), $y_i \in T_i(x)$. Let $z_i \in g_i(x_i, y_i)$. Since g_i is l.s.c., there exists a net $\{z_i^\alpha\}$ such that $z_i^\alpha \to z_i$ and $z_i^\alpha \in g_i(x_i^\alpha, y_i^\alpha) \subseteq C_i(x_i^\alpha)$ and $z_i \in C_i(x_i)$. So $g_i(x_i, y_i) \subseteq C_i(x_i)$, hence H_i is closed. Let $y_i^1, y_i^2 \in H_i(x)$ and $\lambda \in [0, 1]$. Then $y_i^1, y_i^2 \in T_i(x)$, $g_i(x_i, y_i^1) \subseteq C_i(x_i)$ and $g_i(x_i, y_i^2) \subseteq C_i(x_i)$. By (2), we have either

$$g_i(x_i, \lambda y_i^1 + (1 - \lambda) y_i^2) \subseteq g_i(x_i, y_i^1) + C_i(x_i) \subseteq C_i(x_i) + C_i(x_i) \subseteq C_i(x_i)$$

or

$$g_i(x_i, \lambda y_i^1 + (1 - \lambda) y_i^2) \subseteq g_i(x_i, y_i^2) + C_i(x_i) \subseteq C_i(x_i) + C_i(x_i) \subseteq C_i(x_i)$$

and $T_i(x)$ is convex, so $\lambda y_i^1 + (1 - \lambda) y_i^2 \in T_i(x)$. Therefore $H_i(x)$ is convex. By Theorem 2.1 (3), H_i is a compact u.s.c. multivalued map with nonempty closed convex values. By Theorem 3.2, there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ and $\bar{y}_i \in H_i(\bar{x})$ such that $f_i(\bar{x}_i, \bar{y}_i, u_i) \subseteq C_i(\bar{x}_i)$ for all $u_i \in S_i(\bar{x}, \bar{y})$. □

Remark 3.1 Corollary 3.1 holds, if we replayed (2b) by (2b′) for each $(x_i, y_i) \in X_i \times Y_i$, $u_i \multimap f_i(x_i, y_i, u_i)$ is $C_i(x_i)-$quasiconvex with $f_i(x_i, y, x_i) \subseteq C_i(x_i)$.

Theorem 3.3 *For each $i \in I$, suppose that*

(1) $A_i : X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*
(2) $C_i : X_i \multimap Z_i$ *is a closed multivalued map such that* $C_i(x_i)$ *is a proper convex cone and* $\text{int}C_i(x)$ *is nonempty;* $P_i : X_i \multimap Z_i$ *defined by* $P_i(x_i) = Z_i \backslash (-\text{int}C_i(x_i))$ *is an u.s.c. multivalued map;*
(3) $g_i : X_i \times Y_i \times Y_i \multimap Z_i$ *is an u.s.c multivalued map with compact values such that* $g_i(x_i, y_i, y_i) \subseteq C_i(x_i)$*;*
(4) *for each* $(x_i, v_i) \in X_i \times Y_i$, $y_i \multimap g_i(x_i, y_i, v_i)$ *is* $C_i(x_i)-$*quasiconcave, and for each* $(x_i, y_i) \in X_i \times Y_i$, $v_i \multimap g_i(x_i, y_i, v_i)$ *is* $C_i(x_i)-$*quasiconvex.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \nsubseteq -\text{int}C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

Proof For each $i \in I$, let $H_i : X \multimap \overline{T_i(X)}$ be defined by

$$H_i(x) = \{y_i \in T_i(x) \mid g_i(x_i, y_i, v_i) \nsubseteq -\text{int}C_i(x_i) \quad \forall\, v_i \in T_i(x)\}$$

and $Q_i : T_i(x) \multimap T_i(x)$ be defined by

$$Q_i(v_i) = \{y_i \in T_i(x) \mid g_i(x_i, y_i, v_i) \nsubseteq -\text{int}C_i(x_i)\}$$

Suppose there exists a finite set $\{v_i^1, v_i^2, \ldots, v_i^n\}$ in $T_i(x)$ such that $\text{co}\{v_i^1, v_i^2, \ldots, v_i^n\} \nsubseteq \bigcup_{k=1}^n Q_i(v_i^k)$. So we can find a $v_i^\lambda = \lambda_1 v_i^1 + \lambda_2 v_i^2 + \cdots + \lambda_n v_i^n \in \text{co}\{v_i^1, v_i^2, \ldots, v_i^n\}$ where $\lambda_k \geq 0$ for $k = 1, 2, \ldots, n$ and $\sum_{k=1}^n \lambda_k = 1$, but $v_i^\lambda \notin \cup_{k=1}^n Q_i(v_i^k)$. Since $T_i(x)$ is convex, $v_i^\lambda \in T_i(x)$. So $g_i(x_i, v_i^\lambda, v_i^k) \subseteq -\text{int}C_i(x_i)$ for all $k = 1, 2, \ldots, n$. By (3), (4) and Theorem 2.3, there exists $1 \leq j \leq n$ such that

$$g_i(x_i, v_i^\lambda, v_i^j) \subseteq g_i(x_i, v_i^\lambda, v_i^\lambda) + C_i(x_i) \subseteq C_i(x_i) + C_i(x_i) \subseteq C_i(x_i).$$

By (2), $C_i(x_i)$ is a proper cone in Z_i, so $C_i(x_i) \cap (-\text{int}C_(x_i)) = \emptyset$. Then we have $g_i(x_i, v_i^\lambda, v_i^j) \cap (-\text{int}C_i(x_i)) = \emptyset$. This leads to a contradiction. Therefore, Q_i is a KKM map. Let $y_i \in \overline{Q_i(v_i)}$, then there exists a net $\{y_i^\alpha\}_{\alpha \in \wedge}$ in $Q_i(v_i)$ such that $y_i^\alpha \to y_i$. Then $y_i^\alpha \in T_i(x)$ and $g_i(x_i, y_i^\alpha, v_i) \nsubseteq -\text{int}C_i(x_i)$, so $g_i(x_i, y_i^\alpha, v_i) \cap P_i(x_i) \neq \emptyset$. Let $z_i^\alpha \in g_i(x_i, y_i^\alpha, v_i) \cap P_i(x_i)$ and $K_i = \{(x_i, y_i^\alpha, v_i) : \alpha \in \wedge\} \cup \{(x_i, y_i, v_i)\}$, then K_i is compact. By (3) and Theorem 2.1, g_i is closed and $g_i(K_i)$ is compact. Moreover, P_i is closed, so we have $\{z_i^\alpha\}$ has a subnet $\{z_i^{\alpha_\lambda}\}$ such that $z_i^{\alpha_\lambda} \to z_i$ and $z_i \in g_i(x_i, y_i, v_i) \cap P_i(x_i)$. Therefore $g_i(x_i, y_i, v_i) \nsubseteq -\text{int}C_i(x_i)$. Since $T_i(x)$ is closed, $y_i \in T_i(x)$. This shows that $Q_i(v_i)$ is closed. $Q_i(v_i)$ is closed in a compact set $\overline{T_i(X)}$, hence $Q_i(v_i)$ is also compact. Then $\cap_{v_i \in T_i(x)} Q_i(v_i) \neq \emptyset$, then we have $H_i(x) \neq \emptyset$. Let y_i^1, $y_i^2 \in H_i(x)$ and $\lambda \in [0, 1]$. Then $y_i^1, y_i^2 \in T_i(x)$, $g_i(x_i, y_i^1, v_i) \nsubseteq -\text{int}C_i(x_i)$ and $g_i(x_i, y_i^2, v_i) \nsubseteq -\text{int}C_i(x_i)$ for all $v_i \in T_i(x)$. Since $T_i(x)$ is convex, $\lambda y_i^1 + (1 - \lambda) y_i^2 \in T_i(x)$. Suppose that there exist a $\lambda_0 \in [0, 1]$ and $v_i \in T_i(x)$ such that $g_i(x_i, y_i^{\lambda_0}, v_i) \subseteq -\text{int}C_i(x_i)$ where $y_i^{\lambda_0} = \lambda_0 y_i^1 + (1 - \lambda_0) y_i^2$. By (4), either □

$$g_i(x_i, y_i^1, v_i) \subseteq g_i(x_i, y_i^{\lambda_0}, v_i) - C_i(x_i) \subseteq -\text{int}C_i(x_i) - C_i(x_i) \subseteq -\text{int}C_i(x_i)$$

or

$$g_i(x_i, y_i^2, v_i) \subseteq g_i(x_i, y_i^{\lambda_0}, v_i) - C_i(x_i) \subseteq -\text{int}C_i(x_i) - C_i(x_i) \subseteq -\text{int}C_i(x_i).$$

This leads to a contradiction. Therefore $H_i(x)$ is convex. Let $(x,y_i) \in \overline{\mathrm{Gr}H_i}$, then there exists a net $\{(x^\alpha,y_i^\alpha)\}$ in $\mathrm{Gr}H_i$ such that $(x^\alpha,y_i^\alpha) \to (x,y_i)$. So $y_i^\alpha \in T_i(x^\alpha)$ and $g_i(x_i^\alpha,y_i^\alpha,v_i) \nsubseteq -\mathrm{int}C_i(x_i^\alpha)$ for all $v_i \in T_i(x^\alpha)$. By Theorem 2.1, $y_i \in T_i(x)$. And we have $g_i(x_i^\alpha,y_i^\alpha,v_i) \bigcap P_i(x_i^\alpha) \neq \emptyset$ for all $v_i \in T_i(x^\alpha)$. For each $v_i \in T_i(x)$, since T_i is l.s.c., then there exists a net $\{v_i^\alpha\}_{\alpha\epsilon\wedge}$ such that $v_i^\alpha \to v_i$ and $v_i^\alpha \in T_i(x^\alpha)$. Then $g_i(x_i^\alpha,y_i^\alpha,v_i^\alpha) \cap P_i(x_i^\alpha) \neq \emptyset$. Let $z_i^\alpha \in g_i(x_i^\alpha,y_i^\alpha,v_i^\alpha) \cap P_i(x_i^\alpha)$ and $K_i \equiv \{(x_i^\alpha,y_i^\alpha,v_i^\alpha)$: $\alpha\epsilon\wedge\} \cup \{(x_i,y_i,v_i)\}$ which is compact, so $g_i(K_i)$ is compact and g_i is closed by Theorem 2.1. Then $\{z_i^\alpha\}$ has a subnet $\{z_i^{\alpha_\lambda}\}$ such that $z_i^{\alpha_\lambda} \to z_i$. Since g_i and P_i are closed, we have $z_i \in g_i(x_i,y_i,v_i) \cap P_i(x_i) \neq \emptyset$. Therefore $g_i(x_i,y_i,v_i) \nsubseteq -\mathrm{int}C_i(x_i)$ for all $v_i \in T_i(x)$, and then H_i is closed.

By Theorem 2.1 (2), H_i is a compact u.s.c. with nonempty closed convex value.

Define $F\colon X\times Y \multimap X\times Y$ by $F(x,y) = \prod_{i\in I}[A_i(x,y)\times H_i(x)]$. Then F is an u.s.c. with nonempty closed convex value. By Himmelberg fixed theorem, there exists $(\bar{x},\bar{y}) \in X \times Y$ such that $(\bar{x},\bar{y}) \in F(\bar{x},\bar{y})$. There exists $\bar{x}_i \in A_i(\bar{x},\bar{y})$ and $\bar{y}_i \in T_i(\bar{x})$ such that $g_i(\bar{x}_i,\bar{y}_i,v_i) \nsubseteq -\mathrm{int}C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$ □

Corollary 3.4 *Suppose conditions* (2)–(4) *of Theorem* 3.3 *hold, and for each* $i \in I$, *suppose that*

(i) $S_i\colon X \times Y \multimap X_i$ *is a compact continuous multivalued map with nonempty closed convex values*;
(ii) (a) $f_i\colon X_i \times Y_i \times X_i \multimap Z_i$ *is an u.s.c. multivalued map with compact values and concave in the first argument*;
  (b) *for each* $y_i \in Y_i$, *the function* $f_i(\cdot,y_i,\cdot)$ *is weak type II* C_i*-diagonally quasi-convex*;
(iii) $P_i\colon X_i \multimap Z_i$ *defined by* $P_i(x_i) = Z_i\backslash(-\mathrm{int}C_i(x_i))$ *is a concave u.s.c. multivalued map.*

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x},\bar{y})$, $\bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i,\bar{y}_i,u_i) \nsubseteq -\mathrm{int}C_i(\bar{x}_i) \quad \text{for all } u_i \in S_i(\bar{x},\bar{y})$$

and

$$g_i(\bar{x}_i,\bar{y}_i,v_i) \nsubseteq -\mathrm{int}C_i(\bar{x}_i) \quad \text{for all } v_i \in T_i(\bar{x}).$$

Proof For each $i \in I$, define a multivalued map $M_i\colon X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x,y) = \{w_i \in S_i(x,y) \mid f_i(w_i,y_i,u_i) \nsubseteq -\mathrm{int}C_i(w_i) \quad \forall\, u_i \in S_i(x,y)\}.$$

Following the similarly argument as in Theorem 3.3, we can show that $M_i(x,y)$ is nonempty. Let $w_i^1,\ w_i^2 \in M_i(x,y)$ and $\lambda \in [0,1]$, then $w_i^1,\ w_i^2 \in S_i(x,y)$, $f_i(w_i^1,y_i,u_i) \nsubseteq -\mathrm{int}C_i(w_i^1)$ and $f_i(w_i^2,y_i,u_i) \nsubseteq -\mathrm{int}C_i(w_i^2)$ for all $u_i \in S_i(x,y)$. So there exists $z_i^1 \in f_i(w_i^1,y_i,u_i) \cap P_i(w_i^1)$ and $z_i^2 \in f_i(w_i^2,y_i,u_i) \cap P_i(w_i^2)$ for all $u_i \in S_i(x,y)$. Since $S_i(x,y)$ is convex, $\lambda w_i^1 + (1-\lambda)w_i^2 \in S_i(x,y)$. By (ii.a), we have

$$\begin{aligned}\lambda z_i^1 + (1-\lambda)z_i^2 &\in \lambda f_i(w_i^1,y_i,u_i) + (1-\lambda)f_i(w_i^2,y_i,u_i)\\ &\subseteq f_i(\lambda w_i^1 + (1-\lambda)w_i^2,y_i,u_i)\end{aligned}$$

and $\lambda z_i^1+(1-\lambda)z_i^2 \in \lambda P_i(w_i^1)+(1-\lambda)P_i(w_i^2) \subseteq P_i(\lambda w_i^1+(1-\lambda)w_i^2)$. So $\lambda z_i^1+(1-\lambda)z_i^2 \in f_i(\lambda w_i^1 + (1-\lambda)w_i^2,y_i,u_i) \cap P_i(\lambda w_i^1 + (1-\lambda)w_i^2)$. Therefore $f_i(\lambda w_i^1 + (1-\lambda)w_i^2,y_i,u_i) \nsubseteq -\mathrm{int}C_i(\lambda w_i^1 + (1-\lambda)w_i^2)$ for all $u_i \in S_i(x,y)$. Hence $M_i(x,y)$ is convex. Following the same argument as in Theorem 3.3, we see that M_i is closed. Therefore, M_i is a compact u.s.c. multivalued map with nonempty closed convex value by Theorem 2.1 (2). Then by Theorem 3.3 we have the result. □

The following corollary has some applications in the study of Mathematical Programming with equilibrium constraint.

Corollary 3.5 *Suppose conditions* (2) − (4) *of Theorem* 3.3 *hold and for each* $i \in I$,

(i) $S_i : X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values*;

(ii) (a) $f_i : X_i \times Y_i \multimap Z_i$ *is an u.s.c. multivalued map with compact values and convex in the first argument*;

(b) *for each* $x \in X$ *and* $y = (y_i)_{i \in I} \in Y$, *there exists* $w_i \in S_i(x, y)$ *such that* $f_i(w_i, y_i) \nsubseteq -\text{int}C_i(w_i)$;

(iii) $P_i : X_i \multimap Z_i$ *define by* $P_i(x_i) = Z_i \setminus (-\text{int}C_i(x_i))$ *is a concave u.s.c. multivalued map.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $f_i(\bar{x}_i, \bar{y}_i) \nsubseteq -\text{int}C_i(\bar{x}_i)$ and $g_i(\bar{x}_i, \bar{y}_i, v_i) \nsubseteq -\text{int}C_i(\bar{x}_i)$ for all $v_i \in T_i(\bar{x})$

Proof For each $i \in I$, defined $M_i : X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x, y) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i) \nsubseteq (-\text{int}C_i(w_i))\}$$

Following the similar argument as in Corollary 3.4, we can show that M_i is a compact u.s.c. multivalued map with nonempty closed convex values. And then we obtain the result by Theorem 3.3.

Theorem 3.4 *For each* $i \in I$, *suppose that* $B_i : X \multimap Y_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values and conditions (i) and (iii) of Corollary 3.4*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in B_i(x)$ and

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \nsubseteq -\text{int}C_i(\bar{x}_i) \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y}).$$

Proof For each $i \in I$, define a multivalued map $M_i : X \times Y \multimap \overline{S_i(X \times Y)}$ by

$$M_i(x, y) = \{w_i \in S_i(x, y) \mid f_i(w_i, y_i, u_i) \nsubseteq -\text{int}C_i(w_i) \quad \forall u_i \in S_i(x, y)\}.$$

Now, define $F : X \times Y \multimap X \times Y$ by $F(x, y) = \prod_{i \in I}[M_i(x, y) \times B_i(x)]$, then F is an u.s.c. multivalued map with nonempty closed convex values. By Himmelberg fixed point, we have the result.

Corollary 3.6 *For each* $i \in I$, *suppose that conditions (i),(ii) and (iii) of Corollary 3.4 and*

(a) $g_i : X_i \times Y_i \multimap Z_i$ *is a l.s.c. and for each* $x_i \in X_i$, $y_i \multimap g_i(x_i, y_i)$ *is* $C_i(x_i)-$*quasiconcave;*

(b) *for each* $x = (x_i)_{i \in I} \in X$, *there exists* $w_i \in T_i(x)$ *such that* $g_i(x_i, w_i) \nsubseteq -\text{int}C_i(x_i)$;

(c) $T_i : X \multimap Y_i$ *is a compact u.s.c. with closed convex values.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ such that $g_i(\bar{x}_i, \bar{y}_i) \nsubseteq -\text{int}C_i(\bar{x}_i)$ and $f_i(\bar{x}_i, \bar{y}_i, u_i) \nsubseteq -\text{int}C_i(\bar{x}_i)$ for all $u_i \in S_i(\bar{x}, \bar{y})$.

Proof For each $i \in I$, define $H_i : X \multimap \overline{T_i(X)}$ by

$$H_i(x) = \{y_i \in T_i(x) \mid g_i(x_i, y_i) \nsubseteq -\text{int}C_i(x_i)\}$$

Then using the similarly discussion in Theorem 3.3, we have that H_i is a compact u.s.c. with nonempty closed convex values. Therefore, we have the result by Theorem 3.4. □

Remark 3.2 In Corollary 3.4, if we replayed (2b) by (2b′) for each $(x_i, y_i) \in X_i \times Y_i$, $u_i \multimap f_i(x_i, y_i, u_i)$ is $C_i(x_i)-$quasiconvex with $f_i(x_i, y_i, x_i) \subseteq C_i(x_i)$. Then Corollary 3.4 also holds.

Following the similar argument as in Theorem 3.1, we have the following Theorems.

Theorem 3.5 *For each* $i \in I$, *suppose that*

(1) $A_i : X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values*;
(2) $g_i : X_i \times Y_i \times Y_i \multimap Z_i$ *is a compact u.s.c. multivalued map with nonempty closed values and* $g_i(x_i, y_i, y_i) \subseteq C_i(x_i)$;
(3) *for each* $(x_i, v_i) \in X_i \times Y_i$, $y_i \multimap g_i(x_i, y_i, v_i)$ *is concave, and for each* $(x, y) \in X \times Y$, $v_i \multimap g_i(x, y, v_i)$ *is* $C_i(x)$*-quasiconvex.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i \in A_i(\bar{x}, \bar{y}), \bar{y}_i \in T_i(\bar{x})$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \forall\, v_i \in T_i(\bar{x}).$$

Corollary 3.7 *In Theorem* 3.5, *if we replace condition* (1) *and* (2) *by*

(1′) (a) $S_i :\ X \times Y \multimap X_i$ *is a compact continuous multivalued map with nonempty closed convex values;*
(b) $f_i :\ X_i \times Y_i \times X_i \multimap Z_i$ *is an u.s.c. multivalued map with compact values and concave in the first argument;*
(c) *for each* $y_i \in Y_i$, $f_i(\cdot, y_i, \cdot)$ *is strong type II* C_i-*diagonally quasiconvex.*

(2′) $C_i : X_i \multimap Z_i$ *is a closed concave multivalue map.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I, \bar{x}_i \in S_i(\bar{x}, \bar{y}), \bar{y}_i \in T_i(\bar{x})$,

$$f_i(\bar{x}_i, \bar{y}_i, u_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \text{for all } v_i \in T_i(\bar{x}).$$

Following the similar argument as in Corollary 3.2, we have the following Corollary.

Corollary 3.8 *In Theorem* 3.5, *further, if we assume that* C_i *is concave and replace condition* (1) *by*

(1) (a) $S_i : X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*
(b) $f_i :\ X_i \times Y_i \multimap Z_i$ *is an u.s.c. multivalued map with compact values and concave in the first argument; for each* $x \in X$ *and* $y = (y_i)_{i \in I} \in Y$, *there exists* $w_i \in S_i(x, y)$ *such that* $f_i(w_i, y_i) \cap C_i(w_i) \neq \emptyset$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $f_i(\bar{x}_i, \bar{y}_i) \cap C_i(\bar{x}_i) \neq \emptyset$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap C_i(\bar{x}_i) \neq \emptyset \quad \forall\, v_i \in T_i(\bar{x}).$$

Theorem 3.6 *For each $i \in I$, suppose that*

(1) $A_i\colon X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*
(2) $g_i\colon X_i \times Y_i \times Y_i \multimap Z_i$ *is a l.s.c. multivalued map and* $g_i(x_i, y_i, y_i) \subseteq C_i(x_i)$*;*
(3) *for each* $(x_i, v_i) \in X_i \times Y_i$, $y_i \multimap g_i(x_i, y_i, v_i)$ *is convex, and for each* $(x_i, y_i) \in X_i \times Y_i$, $v_i \multimap g_i(x_i, y_i, v_i)$ *is* $C_i(x_i)$*-quasiconvex;*
(4) $P_i\colon X_i \multimap Z_i$ *defined by* $P_i(x_i) = Z_i \setminus (-\mathrm{int}C_i(x_i))$ *is a concave u.s.c. multivalued map.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in A_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap (-\mathrm{int}C_i(\bar{x}_i)) = \emptyset \quad \forall\, v_i \in T_i(\bar{x}).$$

Following the similar argument as in Corollary 3.2, we have the following Corollary.

Corollary 3.9 *In Theorem* 3.6*, further, if we assume C_i is concave and replace condition* (1) *by*

(1′) (a) $S_i : X \times Y \multimap X_i$ *is a compact u.s.c. multivalued map with nonempty closed convex values;*
(b) $f_i : X_i \times Y_i \times X_i \multimap Z_i$ *is a l.s.c. multivalued map and convex in the first argument; for each $x \in X$ and $y = (y_i)_{i \in I} \in Y$, there exists $w_i \in S_i(x, y)$ such that* $f_i(w_i, y_i) \cap -\mathrm{int}C_i(w_i) = \emptyset$.

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in S_i(\bar{x}, \bar{y})$, $\bar{y}_i \in T_i(\bar{x})$, $f_i(\bar{x}_i, \bar{y}_i) \cap (-\mathrm{int}C_i(\bar{x}_i)) = \emptyset$ and

$$g_i(\bar{x}_i, \bar{y}_i, v_i) \cap (-\mathrm{int}C_i(\bar{x}_i)) = \emptyset \quad \forall\, v_i \in T_i(\bar{x}).$$

4 Applications to systems of quasi-saddle point problems and system of quasi-minimax inequalities

In this section, we define systems of quasi-saddle point problems and system of quasi-minimax inequalities.

Let $\varphi_i : X_i \times Y_i \to Z_i$ be a function. We consider the following systems of quasi-saddle point problems.

(SVQSPP): find $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ and $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ such that for each

$$i \in I, \quad \bar{x}_i \in S_i(\bar{x}, \bar{y}), \quad \bar{y}_i \in T_i(\bar{x}),$$

$$\varphi_i(x_i, \bar{y}_i) - \varphi_(\bar{x}_i, \bar{y}_i) \in C_i(\bar{x}_i) \quad \text{ for all } x_i \in S_i(\bar{x}, \bar{y})$$

and

$$\varphi_i(\bar{x}_i, \bar{y}_i) - \varphi_i(\bar{x}_i, y_i) \in C_i(\bar{x}_i) \quad \text{for all } y_i \in T_i(\bar{x}).$$

Theorem 4.1 *For each $i \in I$, suppose that*

(1) (a) $f_i : X_i \times Y_i \to Z_i$ *is a continuous function and affine in the first argument;*
(b) *for each* $x_i \in X_i$, $y_i \to f_i(x_i, y_i)$ *is* $C_i(x_i)$*-quasiconcave; for any finite set* $A = \{x_i^1, x_i^2, \ldots, x_i^n\}$ *in* X_i *and* $x_i \in coA$, *there exists* $1 \le j \le n$ *such that* $f_i(x_i^j, y_i) - f_i(x_i, y_i) \in C_i(x_i)$;

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$f_i(u_i, \bar{y}_i) - f_i(\bar{x}_i, \bar{y}_i) \in C_i(\bar{x}_i) \quad \forall\, u_i \in S_i(\bar{x}, \bar{y})$$

and

$$f_i(\bar{x}_i, \bar{y}_i) - f_i(\bar{x}_i, v_i) \in C_i(\bar{x}_i) \;\; \forall\, v_i \in T_i(\bar{x}).$$

Proof Let $F_i : X_i \times Y_i \times X_i \multimap Z_i$ and $G_i : X_i \times Y_i \times Y_i \multimap Z_i$ be defined by $F_i(x_i, y_i, u_i) = \{f_i(u_i, y_i) - f_i(x_i, y_i)\}$ and $G_i(x_i, y_i, v_i) = \{f_i(x_i, y_i) - f_i(x_i, v_i)\}$. Then by (2a), F_i and G_i are l.s.c. Let $x_i^1,\, x_i^2 \in X_i$ and $\lambda \in [0, 1]$, then by condition (2a), we have

$$\begin{aligned}
&F_i(\lambda x_i^1 + (1-\lambda)x_i^2, y_i, u_i)\\
&\quad = \{f_i(u_i, y_i) - f_i(\lambda x_i + (1-\lambda)x_i^2, y_i)\}\\
&\quad = \{f_i(u_i, y_i) - \lambda f_i(x_i^1, y_i) - (1-\lambda)f_i(x_i^2, y_i)\}\\
&\quad = \{\lambda[f_i(u_i, y_i) - f_i(x_i^1, y_i)] + (1-\lambda)[f_i(u_i, y_i) - f_i(x_i^2, y_i)]\}\\
&\quad \subseteq \lambda\{f_i(u_i, y_i) - f_i(x_i^1, y_i)\} + (1-\lambda)\{f_i(u_i, y_i) - f_i(x_i^2, y_i)\}\\
&\quad = \lambda F_i(x_i^1, y_i, u_i) + (1-\lambda)F_i(x_i^2, y_i, u_i).
\end{aligned}$$

So F_i is convex in the first argument. □

And by condition (1b), we can obtain that F_i is strong type I C_i-diagonally quasiconvex. Note that $G_i(x_i, y_i, y_i) = \{f_i(x_i, y_i) - f_i(x_i, y_i)\} = \{0\} \subseteq C_i(x_i)$. By condition (2b), $y_i \to f_i(x_i, y_i) - f_i(x_i, v_i)$ is $C_i(x_i)-$ quasiconcave and $v_i \to f_i(x_i, y_i) - f_i(x_i, v_i)$ is $C_i(x_i)-$ quasiconvex. Then by Corollary 3.1, there exists there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$f_i(u_i, \bar{y}_i) - f_i(\bar{x}_i, \bar{y}_i) \in C_i(\bar{x}_i) \quad \forall\, u_i \in S_i(\bar{x}, \bar{y})$$

and

$$f_i(\bar{x}_i, \bar{y}_i) - f_i(\bar{x}_i, v_i) \in C_i(\bar{x}_i) \quad \forall\, v_i \in T_i(\bar{x}).$$

Theorem 4.2 *In Theorem* 4.1, *if we let* $Z_i = \mathbb{R}$ *and* $y_i \to f_i(x_i, y_i)$ *is quasiconcave, then there exists* $\bar{x} = (\bar{x}_i)_{i \in I} \in X$ *and* $\bar{y} = (\bar{y}_i)_{i \in I} \in Y$ *with* $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ *and* $\bar{y}_i \in T_i(\bar{x})$ *such that*

$$\min_{u_i \in S_i(\bar{x}, \bar{y})} \max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i) = f_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i).$$

Proof $C_i(x_i) = [0, \infty)$ for all $x \in X$. Then by Theorem 4.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ with $\bar{x}_i \in S_i(\bar{x}, \bar{y})$ and $\bar{y}_i \in T_i(\bar{x})$ such that

$$f_i(u_i, \bar{y}_i) \ge f_i(\bar{x}_i, \bar{y}_i) \;\; \text{for all } u_i \in S_i(\bar{x}, \bar{y})$$

and

$$f_i(\bar{x}_i, \bar{y}_i) \geq f_i(\bar{x}_i, v_i) \text{ for all } v_i \in T_i(\bar{x}).$$

That is,

$$f_i(\bar{x}_i, \bar{y}_i) = \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, \bar{y}_i)$$

and

$$f_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} f_i(\bar{x}_i, v_i).$$

Then

$$f_i(\bar{x}_i, \bar{y}_i) = \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, \bar{y}_i) \leq \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i)$$

and

$$f_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} f_i(\bar{x}_i, v_i) \geq \min_{u_i \in S_i(\bar{x}, \bar{y})} \max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i).$$

So

$$\min_{u_i \in S_i(\bar{x}, \bar{y})} \max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i) \leq f_i(\bar{x}_i, \bar{y}_i) \leq \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i).$$

And

$$\min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i) \leq f_i(\bar{x}_i, v_i) \leq \max_{v_i \in T_i(\bar{x})} f_i(\bar{x}_i, v_i) = f_i(\bar{x}_i, \bar{y}_i).$$

$$\max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i) \geq f_i(u_i, \bar{y}_i) \geq \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, \bar{y}_i) = f_i(\bar{x}_i, \bar{y}_i).$$

So we have

$$\max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i) \leq f_i(\bar{x}_i, \bar{y}_i) \leq \min_{u_i \in S_i(\bar{x}, \bar{y})} \max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i).$$

Therefore

$$\min_{u_i \in S_i(\bar{x}, \bar{y})} \max_{v_i \in T_i(\bar{x})} f_i(u_i, v_i) = f_i(\bar{x}_i, \bar{y}_i) = \max_{v_i \in T_i(\bar{x})} \min_{u_i \in S_i(\bar{x}, \bar{y})} f_i(u_i, v_i).$$ □

5 Applications to mathematical program with equailibrium constraint, semi-infinite and bilevel problems

Theorem 5.1 *In Corollary* 3.1, *in addition, let* L *be a real t.v.s. and* $h: X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:* (P_1) Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid \forall\, i \in I,\ x_i \in S_i(x, y), y_i \in T_i(x), f_i(x_i, y_i, u_i) \subseteq C_i(x_i)\ \forall\ u_i \in S_i(x, y),$ *and* $g_i(x_i, y_i, v_i) \subseteq C_i(x_i)\ \forall\ v_i \in T_i(x)\}$.

Proof By Corollary 3.1, we have that $K \neq \emptyset$. Let $(x, y) \in \overline{K}$, then there exists a net $\{(x^\alpha, y^\alpha)\}$ in K such that $(x^\alpha, y^\alpha) \to (x, y)$. So for each $i \in I$, $x_i^\alpha \in S_i(x^\alpha, y^\alpha)$, $y_i^\alpha \in T_i(x^\alpha), f_i(x_i^\alpha, y_i^\alpha, u_i) \subseteq C_i(x_i^\alpha)$ for all $u_i \in S_i(x^\alpha, y^\alpha)$ and $g_i(x_i^\alpha, y_i^\alpha, v_i) \subseteq C_i(x_i^\alpha)$ for all $v_i \in T_i(x^\alpha)$. Since S_i and T_i are closed, $x_i \in S_i(x, y)$ and $y_i \in T_i(x)$. Let $u_i \in S_i(x, y)$. Since S_i is l.s.c., there exists a net $\{u_i^\alpha\}$ such that $u_i^\alpha \to u_i$ and $u_i^\alpha \in S_i(x^\alpha, y^\alpha)$. Let

$z_i \in f_i(x_i, y_i, u_i)$. Since f_i is l.s.c., there exists a net $\{z_i^\alpha\}$ such that $z_i^\alpha \to z_i$ and $z_i^\alpha \in f_i(x_i^\alpha, y_i^\alpha, u_i^\alpha) \subseteq C_i(x_i^\alpha)$. Since C_i is closed, $z_i \in C_i(x_i)$. So $f_i(x_i, y_i, u_i) \subseteq C_i(x_i)$ for all $u_i \in S_i(x, y)$. Similarly, by the same way we have $g_i(x_i, y_i, v_i) \subseteq C_i(x_i)$ for all $v_i \in T_i(x)$. Therefore $(x, y) \in K$, i.e. K is closed in a compact set $\prod_{i\in I} \overline{S_i(X \times Y)} \times \overline{T_i(X)}$, hence K is also compact. And since h is u.s.c. with compact valued, it follows Theorem 2.4 that P_1 has a solution. □

Theorem 5.2 *In Corollary* 3.3, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:*

(MPEC$_1$) Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I$, $x_i \in S_i(x, y)$, $y_i \in T_i(x), g_i(x_i, y_i) \subseteq C_i(x_i)$, *and* $f_i(x_i, y_i, u_i) \subseteq C_i(x_i)$ *for all* $u_i \in S_i(x, y)\}$.

Proof By Corollary 3.3, we have that $K \neq \emptyset$. With the similarly discussion in Theorem 5.1, we can show that K is compact, hence the conclusion is true. □

Corollary 5.1 *In Corollary* 3.3, *if we replace condition* (1) *by*

(1′) $\varphi_i : X_i \times X_i \multimap Z_i$ *is a l.s.c. multivalued map and convex in the first argument;* φ_i *is strong type I* C_i*-diagonally quasiconvex*

and h be the same as in Theorem 5.2. Then there exists a solution of the problem:

(SIP_1) *Min* $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I$, $x_i \in S_i(x, y)$, $y_i \in T_i(x)$, $g_i(x_i, y_i) \subseteq C_i(x_i)$, *and* $\varphi_i(x_i, u_i) \subseteq C_i(x_i)$ *for all* $u_i \in S_i(x, y)\}$.

Proof Define $f_i : X_i \times Y_i \times X_i \multimap Z_i$ by $f_i(x_i, y_i, u_i) = \varphi_i(x_i, u_i)$. Then f_i is l.s.c. and convex in the first argument and $f(\cdot, y, \cdot)$ is strong type I C_i-diagonally quasiconvex. By Theorem 5.2, we have the conclusion. □

Applying Corollary 3.1, we have the following existence theorem of mathematical program with equilibrium constraint.

Theorem 5.3 *In Corollary* 3.2, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there is a solution of the problem:*

($MPEC_2$) Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I$, $x_i \in S_i(x, y)$, $y_i \in T_i(x)$, $f_i(x_i, y_i) \subseteq C_i(x_i)$, *and* $g_i(x_i, y_i, v_i) \subseteq C_i(x_i)$ *for all* $v_i \in T_i(x)\}$.

For the special case of Theorem 5.3, we have the following Corollary.

Corollary 5.2 *Let* S_i, T_i, h *be the same as in Theorem 5.3,* $C_i(x_i) = [0, \infty)$ *and for each* $i \in I$, *suppose that*

(1) (a) $f_i\colon X_i \times Y_i \to \mathbb{R}$ *is an affine continuous function;*
    (b) *for each* $x = (x_i)_{i\in I} \in X$ *and* $y = (y_i)_{i\in I} \in Y$, *there exists* $w_i \in S_i(x, y)$ *such that* $f_i(w_i, y_i) \geq 0$;
(2) $g_i\colon X_i \times Y_i \times Y_i \to \mathbb{R}$ *is a continuous function such that* $g_i(x_i, y_i, y_i) \geq 0$;
(3) *for each* $(x_i, v_i) \in X_i \times Y_i$, $y_i \to g_i(x_i, y_i, v_i)$ *is quasiconcave, and for each* $(x_i, y_i) \in X_i \times Y_i$, $v_i \to g_i(x_i, y_i, v_i)$ *is quasiconvex.*

Then there is a solution of the problem:

(MPEC$_3$) Min $h(K)$, where $K = \{(x,y) \in X \times Y \mid$ for each $i \in I$, $x_i \in S_i(x,y)$, $y_i \in T_i(x)$, $f_i(x_i,y_i) \geq 0$, and $g_i(x_i,y_i,v_i) \geq 0$ for all $v_i \in T_i(x)\}$.

As an application of Corollary 5.2, we establish the existence theorem of bilevel problem.

Corollary 5.3 *In Corollary* 5.2, *we replace conditions* (2) *and* (3) *by*

(2′) $\varphi_i\colon X_i \times Y_i \to \mathbb{R}$ *is a continuous function;*
(3′) *for each* $x_i \in X_i$, $y_i \to \varphi_i(x_i,y_i)$ *is quasiconvex.*

Then there exist a solution of the following Bilevel Problems: (BL) Min $h(K)$, where $K = \{(x,y) \in X \times Y \mid$ for each $i \in I$, $x_i \in S_i(x,y)$, $y_i \in T_i(x)$, $f_i(x_i,y_i) \geq 0$, and $g_i(x_i,y_i) \leq g_i(x_i,v_i)$ for all $v_i \in T_i(x)\}$.

Proof Let $Z_i = \mathbb{R}$, $C_i = [0,\infty)$ and $g_i(x_i,y_i,v_i) = \varphi_i(x_i,v_i) - \varphi_i(x_i,y_i)$, then $g_i(x_i,y_i,y_i) = 0 \in C_i(x_i)$. Moreover, $g_i(x_i,\cdot,v_i)$ is quasiconcave and $g_i(x_i,y_i,\cdot)$ is quaisconvex. Then we obtain the conclusion by Corollary 5.2. □

By Corollary 3.1, we have the following theorem.

Theorem 5.4 *In Corollary* 3.4, *in addition, let L be a real t.v.s. and* $h\colon X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:* (P_2) Min $h(K)$, *where* $K = \{(x,y) \in X \times Y \mid \forall\, i \in I$, $x_i \in S_i(x,y), y_i \in T_i(x)$, $f_i(x_i,y_i,u_i) \nsubseteq -\text{int}C_i(x_i)\ \forall\, u_i \in S_i(x,y)$, *and* $g_i(x_i,y_i,v_i) \nsubseteq -\text{int}C_i(x_i)\ \forall\, v_i \in T_i(x)\}$.

Theorem 5.5 *In Corollary* 3.6, *in addition, let L be a real t.v.s. and* $h: X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:*

(MPEC$_4$) Min $h(K)$, *where* $K = \{(x,y) \in X \times Y \mid$ *for each* $i \in I$, $x_i \in S_i(x,y)$, $y_i \in T_i(x)$, $g_i(x_i,y_i) \nsubseteq -\text{int}C_i(x_i)$, *and* $f_i(x_i,y_i,u_i) \nsubseteq -\text{int}C_i(x_i)$ *for all* $u_i \in S_i(x,y)\}$.

By Corollary 3.6, we have the existence theorem of semi-infinite problem.

Corollary 5.4 *In Corollary* 3.6, *if we replace condition (ii.a) and (ii.b) of Corollary 3.4 by*

(2′) $\varphi_i : X_i \times X_i \multimap Z_i$ *is a continuous multivalued map and concave in the first argument;* φ_i *is weak type II* C_i*-diagonally, quasiconvex*

and h be the same as in Theorem 5.5. Then there exists a solution of the problem:

(SIP$_2$) Min $h(K)$, where $K = \{(x,y) \in X \times Y \mid$ for each $i \in I$, $x_i \in S_i(x,y)$, $y_i \in T_i(x)$, $g_i(x_i,y_i) \nsubseteq -\text{int}C_i(x_i)$, and $\varphi_i(x_i,u_i) \nsubseteq -\text{int}C_i(x_i)$ for all $u_i \in S_i(x,y)\}$.

Proof Define $f_i\colon X_i \times Y_i \times X_i \multimap Z_i$ by $f_i(x_i,y_i,u_i) = \varphi_i(x_i,u_i)$. Then f_i is an u.s.c. multivalued map and concave in the first argument. Moreover, $f(\cdot,y,\cdot)$ is weak type II C_i-diagonally quasiconvex. By Theorem 5.5, we have the conclusion. □

Theorem 5.6 *In Corollary* 3.5, *in addition, let L be a real t.v.s. and* $h\colon X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there is a solution of the problem:*

(MPEC$_5$) Min $h(K)$, *where* $K = \{(x,y) \in X \times Y \mid$ *for each* $i \in I$, $x_i \in S_i(x,y)$, $y_i \in T_i(x)$, *and* $f_i(x_i,y_i) \nsubseteq -\text{int}C_i(x_i), g_i(x_i,y_i,v_i) \nsubseteq -\text{int}C_i(x_i)$ *for all* $v_i \in T_i(x)\}$.

Theorem 5.7 *In Corollary* 3.7, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:*

(P_3) Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid \forall\, i \in I,\ x_i \in S_i(x, y), y_i \in T_i(x),\ f_i(x_i, y_i, u_i) \cap C_i(x_i) \neq \emptyset\ \forall\, u_i \in S_i(x, y),$ *and* $g_i(x_i, y_i, v_i) \cap C_i(x_i) \neq \emptyset\ \forall\, v_i \in T_i(x)\}$.

Theorem 5.8 *In Corollary* 3.9, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there exists a solution of the problem:*

$(MPEC_6)$ Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x),\ g_i(x_i, y_i) \cap C_i(x_i) \neq \emptyset,$ *and* $f_i(x_i, y_i, u_i) \cap C_i(x_i) \neq \emptyset$ *for all* $u_i \in S_i(x, y)\}$.

Theorem 5.9 *In Corollary* 3.8, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact values. Then there is a solution of the problem:*

$(MPEC_7)$ Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x),$ *and* $f_i(x_i, y_i) \cap C_i(x_i) \neq \emptyset, g_i(x_i, y_i, v_i) \cap C_i(x_i) \neq \emptyset$ *for all* $v_i \in T_i(x)\}$.

Theorem 5.10 *In Corollary* 3.8, *in addition, let L be a real t.v.s. and* $h : X \times Y \multimap L$ *be an u.s.c. multivalued map with compact valued. Then there is a solution of the problem:*

$(MPEC_8)$ Min $h(K)$, *where* $K = \{(x, y) \in X \times Y \mid$ *for each* $i \in I,\ x_i \in S_i(x, y),\ y_i \in T_i(x),$ *and* $f_i(x_i, y_i) \cap (-\mathrm{int} C_i(x_i)) = \emptyset, g_i(x_i, y_i, v_i) \cap (-\mathrm{int} C_i(x_i)) = \emptyset$ *for all* $v_i \in T_i(x)\}$.

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