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Mathematical programming with system of equilibrium constraints

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Abstract In this paper, we study the mathematical program with system of equilibrium constraints. This problem contains bilevel program with system of equilibrium constraints, semi-infinite program with system of equilibrium constraints, mathematical program with Nash equilibrium constraints, mathematical program with system of mixed variational like inequalities constraints. We establish the existence theorems of mathematical program with system of equilibrium constraints under various assumptions.

Keywords Mathematical program (resp. bilevel problem, semi-infinite problem) with system of equilibrium constraints · Concave (resp. convex) multivalued map · Upper (resp. lower) semicontinuous multivalued map

1 Introduction

Let I be any index set. For each $i \in I$, let X_i be a nonempty subset of a topological space E_i, Y_i be a nonempty subset of a topological vector space (in short t.v.s.) V_i, $X = \Pi_{i\in I} X_i$, $Y = \Pi_{i\in I} Y_i$, $f_i : X \times Y_i \times Y_i \to \mathbb{R}$, $h : X \times Y \to \mathbb{R}$ and $g_i : X_i \times Y \to \mathbb{R}$ be functions, $T_i : X \multimap Y_i$ be multivalued map. In this paper, we study the mathematical program with system of equilibrium constraints (MPSEC) of type I.
MPSEC I: $\min_{(x,y)} h(x,y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x_i, y) \geq 0$ and

$$f_i(x, y_i, v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and all } i \in I.$$

If $f_i(x, y_i, v_i) = \varphi_i(x, v_i) - \varphi_i(x, y_i)$, where $\varphi_i : X \times Y_i \to \mathbb{R}$ is a function, then the MPSEC will be reduced to the bilevel problem with system of equilibrium constraints (BLSEC).

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BLSEC: $\min_{(x,y)} h(x,y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x_i, y) \geq 0$ and

$$y_i \text{ is a solution of the problem } Q(x): \min_{v_i \in T_i(x)} \varphi_i(x, v_i) \quad \text{for all } i \in I.$$

If $f_i(x, y_i, v_i) = \varphi_i(x, v_i)$ for all $x \in X$, $y_i \in Y_i$ and $v_i \in Y_i$, then the MPSEC will be reduced to the semi-infinite program with system of equilibrium constraints (SIPSEC):
SIPSEC: $\min_{(x,y)} h(x,y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I}$, $g_i(x_i, y) \geq 0$, $y_i \in T_i(x)$,

$$\text{and } \varphi_i(x, v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

If $f_i(x, y_i, v_i) = \langle F_i(x), \eta_i(y_i, v_i)\rangle + p_i(v_i) - p_i(y_i)$, where $\eta_i\colon Y_i \times Y_i \to Y_i, p_i\colon Y_i \to \mathbb{R}$, are functions $F_i\colon X \to Y_i^*$, where Y_i^* is the dual space of Y_i and $\langle\cdot,\cdot\rangle$ be the dual pair between Y_i and Y_i^*, then the MPSEC will be reduced to the mathematical program with system of mixed variational-like inequalities constraints (MPSMVLI):
MPSMVLI: $\min_{(x,y)} h(x,y), x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I} \in Y, y_i \in T_i(x), g_i(x_i, y) \geq 0$ and

$$\langle F_i(x), \eta_i(y_i, v_i)\rangle + p_i(v_i) - p_i(y_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

If $p_i(y_i) = 0$ for all $y_i \in Y_i$ and for all $i \in I$. Then the MPSMVLI will be reduced to the mathematical program with system of variational-like inequalities constraints.
MPSVLI: $\min_{(x,y)} h(x,y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x_i, y) \geq 0$

$$\text{and } \langle F_i(x), \eta_i(y_i, v_i)\rangle \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

If I is a singleton, $f{:}X \times Y \times Y \to \mathbb{R}$, $g{:}X \times Y \to \mathbb{R}$ and $\varphi\colon X \times Y \to \mathbb{R}$ are functions and $T: x \to Y$ are multivalued maps. Then the MPSEC will be reduced to the problem:

$$\text{MPEC: } \min_{(x,y)} h(x,y) \text{ such that } g(x,y) \geq 0 \text{ and } f(x,y,v) \geq 0 \text{ for all } v \in T(x);$$

BLSEC will be reduced to the problem:

$$\text{BL: } \min_{(x,y)} h(x,y) \text{ such that } g(x,y) \geq 0 \text{ and } y \text{ is a solution of } Q(x): \min_{t \in T(x)} \varphi(x,t);$$

SIPEC will be reduced to the problem:

$$\text{SIP: } \min_{(x,y)} h(x,y) \text{ such that } g(x,y) \geq 0 \text{ and } \varphi(x,v) \geq 0 \text{ for all } v \in T(x).$$

We also study the mathematical problem with systems of equilibrium constraints of type II.
MPSEC II: $\min_{x,y} h(x,y)$ such that $x \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x, y_i) \geq 0$,

$$f_i(x, y, v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I,$$

where $g_i\colon X \times Y_i \to \mathbb{R}$ and $f_i\colon X \times Y \times Y_i \to \mathbb{R}$ are functions.
If $f_i(x, y, v_i) = \varphi_i(y, v_i) - \varphi_i(y, y_i)$. Then the MPSEC II will be reduced to the mathematical program with Nash equilibrium constraints:
MPNEC: $\min_{(x,y)} h(x,y)$ such that $x \in X$, $y = (y_i)_{i\in I}$, $y_i \in T_i(x)$, $g_i(x, y_i) \geq 0$ and

$$\varphi_i(y, v_i) \geq \varphi_i(y, y_i) \quad \text{for all } v_i \in T_i(x).$$

MPEC, SIP; and BL represent three important classes of optimization problems which have been investigated in a large number of papers and books (see, e.g [2, 3, 8–11] and references there in). These papers mainly deal with the optimal conditions and numerical methods used to solve MPEC, SIP, and BL. Typically the existence of a feasible point is tacitly assumed. The aim of this paper is to establish the sufficient conditions for the existence of the feasible points of MPSEC and the solution of this type of problem. We investigate under what assumptions that MPSEC has a solution. The main tools of this paper are maximal element theorem for a family of multi-valued maps and Himmelberg fixed point theorem. Our approach are different from [7]. Since MPSEC contains many problems, as special cases, our results contain many existence results of the problems which are the special cases of mathematical program with system of equilibrium constraints.

2 Preliminaries

Let $T : X -\circ Y$ be a multivalued map from a space X to another space Y. By $GrT = \{(x, y) \in X \times Y : x \in X, y \in T(x)\}$ will denote the graph of T. The inverse T^- of T is the multivalued map defined by $x \in T^-(y)$ if and only $y \in T(x)$.

Let X and Y be topological spaces (in short t.s.). A multivalued map $T: X -\circ Y$ is said to be upper semicontinuous (in short u.s.c.) (resp. lower semicontinuous, in short l.s.c.) at $x \in X$, if for every open set U in Y with $T(x) \subseteq U$ (resp. $T(x) \cap U \neq \emptyset$), there exists an open neighborhood $V(x)$ of x such that $T(x') \subseteq U$ (resp. $T(x') \cap U \neq \emptyset$) for all $x' \in V(x)$; T is said to be u.s.c. (resp. l.s.c.) on X if T is u.s.c. (resp. l.s.c.) at every point of X; T is continuous at x if T is both u.s.c. and l.s.c. at x ; T is said to be closed if $Gr\ T$ is a closed subset of $X \times Y$; T is said to be compact if there exists a compact subset K of Y such that $T(X) \subseteq K$. Let $A \subseteq X$, by $\bar{A}$ will denote the closure of A.

The following theorems and lemma are needed in this paper.

Theorem 2.1 (Himmelberg [6]) *Let X be a convex subset of a locally convex t.v.s. and D be a nonempty compact subset of X. Let $T : X -\circ D$ be an u.s.c. multivalued map such that for each $x \in X$, $T(x)$ is a nonempty closed convex subset of D. Then there exists a point $\bar{x} \in D$ such that $\bar{x} \in T(\bar{x})$.*

Theorem 2.2 [1] *Let X and Y be Hausdorff topological spaces and $T : X -\circ Y$ be a multivalued map.*

(1) If Y is compact and T is closed, then T is u.s.c;
(2) If T is u.s.c. and for each $x \in X$, $T(x)$ is a closed set, then T is closed;
(3) If X is compact and T is u.s.c. with compact values, then $T(X)$ is compact.

Lemma 2.1 [12] *Let X and Y be Hausdorff topological spaces and $T : X -\circ Y$ be a multivalued map and $x \in X$, then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$, and any net $\{x_\alpha\}$, $x_\alpha \to x$, there is a net $\{y_\alpha\}$ such that $y_\alpha \in T(x_\alpha)$ and $y_\alpha \to y$.*

Definition 2.1 Let X and Y be vector spaces and $T : X -\circ Y$ be a multivalued map.

(1) T is concave if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$\lambda T(x_1) + (1 - \lambda)T(x_2) \subset T(\lambda x_1 + (1 - \lambda)x_2);$$

(2) T is convex if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,

$$T(\lambda x_1 + (1 - \lambda)x_2) \subseteq \lambda T(x_1) + (1 - \lambda)T(x_2).$$

Let A be a subset of a t.v.s. E, co$A$ will denote the convex hull of A.

Definition 2.2 Let X be a convex subset of a t.v.s. and a multivalued map $T : X - \circ X$ is called a KKM mapping if for any finite subset N of X;

$$(\mathrm{co}N) \subseteq T(N) = \cup\{T(x) : x \in N\}.$$

Theorem 2.3 [4] *Let E be a Hausdorff t.v.s., Y be a convex subset of E, X be a nonempty subset of Y, $T : X - \circ Y$ be a KKM map. Suppose that for each $x \in X$, $T(x)$ is closed and there exists $x_0 \in X$ such that $T(x_0)$ is compact. Then $\cap_{x\in X} T(x) \neq \emptyset$.*

3 Mathematical programming with systems of equilibrium constraints of type I

In this section, we study the following mathematical programming with systems of equilibrium constraints of type I:

$$\min_{(x,y)} h(x,y), (x,y) \in M_i \quad \text{for all } i \in I, \tag{1}$$

where $M_i = \{(x,y) \in X \times Y : x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I} \in Y, y_i \in T_i(x), g_i(x_i,y) \geq 0, f_i(x,y_i,v_i) \geq 0$ for all $v_i \in T_i(x)\}$.

Theorem 3.1 *Let I be any index set. For each $i \in I$, let X_i be a nonempty compact convex subset of a Hausdorff locally convex t.v.s. E_i, Y_i be a nonempty closed convex subset of a locally convex t.v.s. V_i. Let $Y = \Pi_{i\in I} Y_i$, $X = \Pi_{i\in I} X_i$, T_i:$X - \circ Y_i$ be a continuous multivalued map with nonempty compact convex values. Let $f_i : X \times Y_i \times Y_i \to \mathbb{R}$ and $g_i : X_i \times Y \to \mathbb{R}$ be functions satisfying the following conditions:*

(1) $f_i : X \times Y_i \times Y_i \to \mathbb{R}$ is an u.s.c. function;
(2) for each $(x,y_i) \in X \times Y_i$, $f_i(x,y_i,y_i) \geq 0$ and for each $(x,y_i) \in X \times Y_i$, $v_i \to f_i(x,y_i,v_i)$ is quasiconvex, and for each $(x,v_i) \in X \times Y_i$, $y_i \to f_i(x,y_i,v_i)$ is quasiconcave;
(3) $g_i : X_i \times Y \to \mathbb{R}$ is an u.s.c. function; and
(4) for each fixed $y \in Y$, $x_i \to g_i(x_i,y)$ is quasiconcave; and for each $y \in Y$, there exists $w_i \in X_i$ such that $g_i(w_i,y) \geq 0$.

Then there exists $\bar{x} \in X$, $\bar{y} \in (\bar{y}_i)_{i\in I} \in Y = \Pi_{i\in I} Y_i$ such that $\bar{y}_i \in T_i(\bar{x})$, $g_i(\bar{x}_i,\bar{y}) \geq 0$ and $f_i(\bar{x},\bar{y}_i,v_i) \geq 0$ for all $v_i \in T_i(\bar{x})$ and for all $i \in I$.

Proof Let $A_i : Y - \circ X_i$ be defined by

$$A_i(y) = \{w_i \in X_i : g_i(w_i,y) \geq 0\},$$

where A_i is closed. Indeed, if $(x_i,y) \in \overline{GrA_i}$, then there exists a net $\{(x_i^\alpha, y^\alpha)\}$ in $\mathrm{Gr}A_i$ such that $(x_i^\alpha, y^\alpha) \to (x_i,y)$. One has $x_i^\alpha \in X_i$, $g_i(x_i^\alpha, y^\alpha) \geq 0$. Since X_i is closed and g_i is u.s.c, $x_i \in X_i$ and $g_i(x_i,y) \geq 0$. Therefore, $(x_i,y) \in GrA_i$ and A_i is closed. But $A_i(Y) \subseteq X_i$ and X_i is compact, it follows that $A_i : Y - \circ X$ is u.s.c. As A_i is closed, $A_i(y)$ is a closed set for each $y \in Y$. By assumption, $A_i(y)$ is nonempty. Since for each $y \in Y$, $w_i \to g_i(w,y)$ is quasiconcave and X_i is a convex set, $A_i(y)$ is convex for each $y \in Y$. For each $x \in X$, let $Q_i(x) : T_i(x) - \circ T_i(x)$ be defined by

$$Q_i(x)(v_i) = \{y_i \in T_i(x) : f_i(x,y_i,v_i) \geq 0\}.$$

Then $Q_i(x) : T_i(x) - \circ T_i(x)$ is a KKM map. Ineeed, if $Q_i(x)$ is not a KKM map, then there exists a finite subset $\{v_i^1, v_i^2, \ldots, v_i^n\}$ in $T_i(x)$ such that $\mathrm{co}\{v_i^1, v_i^2, \ldots, v_i^n\} \not\subseteq$

$\cup_{j\in 1}^{n} Q_i(x)(v_i^j)$. Hence there exists $v_i \in \text{co}\{v_i^1, v_i^2, \ldots, v_i^n\}$ such that $v_i \notin Q_i(x)(v_i^j)$ for all $j = 1,2,\ldots,n$. But $v_i^j \in T_i(x)$ and $T_i(x)$ is convex, we see $v_i \in T_i(x)$. Therefore, $f_i(x, v_i, v_i^j) < 0$. Since $u_i \to f_i(x, y_i, u_i)$ is quasiconvex,

$$f_i(x, v_i, v_i) \le \max\{f_i(x, v_i, v_i^1), f_i(x, v_i, v_i^2), \ldots, f_i(x, v_i, v_i^n)\} < 0.$$

This contradicts to $f_i(x, y_i, y_i) \ge 0$ for all $(x, y_i) \in X \times Y_i$. This shows that for each $x \in X$, $Q_i(x) : T_i(x) -\circ T_i(x)$ is a KKM map.
Since for each $x \in X$, $T_i(x)$ is closed and $f_i \colon X \times Y_i \times Y_i \to \mathbb{R}$ is u.s.c. It is easy to see that $Q_i(x)(v_i)$ is a closed subset of $T_i(x)$. But $T_i(x)$ is compact, therefore $Q_i(x)(v_i)$ is a compact set. Then by Theorem 2.3 that $\cap_{v_i\in T_i(x)} Q_i(x)(v_i) \ne \emptyset$. Let $y_i \in \cap_{v_i\in T_i(x)} Q_i(x)(v_i)$, then $y_i \in T_i(x)$ and $f_i(x, y_i, v_i) \ge 0$ for all $v_i \in T_i(x)$.

Let $B_i : X \multimap Y_i$ be defined by

$$B_i(x) = \{y_i \in T_i(x) : f_i(x, y_i, v_i) \ge 0 \text{ for all } v_i \in T_i(x)\}.$$

This shows that $B_i(x) \ne \emptyset$ for all $x \in X$ and $i \in I$. $B_i : X -\circ Y_i$ is closed. Indeed, if $(x, y_i) \in \overline{GrB_i}$, then there exists a net $(x^\alpha, y_i^\alpha) \in GrB_i$ such that $(x^\alpha, y_i^\alpha) \to (x, y_i)$. One has $y_i^\alpha \in T_i(x^\alpha)$ and $f_i(x^\alpha, y_i^\alpha, v_i) \ge 0$ for all $v_i \in T_i(x^\alpha)$. Let $v_i \in T_i(x)$. Since $T_i : X -\circ Y_i$ is l.s.c., there exists a net $\{v_i^\alpha\}$ in $T_i(x^\alpha)$ such that $v_i^\alpha \to v_i$. Since T_i is an u.s.c. multivalued map with closed values, it follows from Theorem 2.2 that T_i is closed and $y_i \in T_i(x)$. We also have $f_i(x^\alpha, y_i^\alpha, v_i^\alpha) \ge 0$. Since f_i is u.s.c., $f_i(x, y_i, v_i) \ge 0$. This shows that $(x, y_i, v_i) \in GrB_i$ and B_i is closed. By the assumption that X is compact and $T_i : X -\circ Y_i$ is an u.s.c. multivalued map with nonempty compact values, it follows from Theorem 2.2 that $T_i(X)$ is compact. But $B_i(X) \subseteq T_i(X)$, then by Theorem 2.2 $B_i : X -\circ Y$ is an u.s.c. multivalued map. Since B_i is closed, $B_i(x)$ is a closed set for each $x \in X$. Let $A : Y -\circ X$ and $B : X -\circ Y$ be defined by $A(y) = \Pi_{i\in I} A_i(y)$ and $B(x) = \Pi_{i\in I} B_i(x)$, then by Lemma 3 [5] that A and B are compact u.s.c. multivalued map with nonempty closed convex values. Let $F : X \times Y -\circ X \times Y$ be defined by $F(x, y) = A(y) \times B(x)$. Again by Lemma 3 [5] that F is a compact u.s.c. multivlaued map with nonempty closed convex values. Then by Himmelberg fixed point theorem that there exists $(\bar{x}, \bar{y}) \in F(\bar{x}, \bar{y})$. That is $\bar{x} \in A(\bar{y})$ and $\bar{y} \in T(\bar{x})$. Therefore, $\bar{x} = (x_i)_{i\in I} \in X$, $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$, $\bar{y}_i \in T_i(\bar{x})$, $g_i(\bar{x}_i, \bar{y}) \ge 0$ and $f_i(\bar{x}, \bar{y}_i, v_i) \ge 0$ for all $v_i \in T_i(\bar{x})$ and for all $i \in I$.

Theorem 3.2 *In Theorem 3.1, if we assume further that $h\colon X \times Y \to \mathbb{R}$ is a l.s.c. function. Then there exists a solution of the program:*
$\min_{(x,y)} h(x, y)$ *such that* $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$ $g_i(x_i, y) \ge 0$ *and* $f_i(x, y_i, v_i) \ge 0$ *for all* $v_i \in T_i(x)$ *and for all* $i \in I$.

Proof For each $i \in I$, M_i is a closed set for each $i \in I$. Indeed, if $(x, y) \in \overline{M_i}$, then there exists a net $(x^\alpha, y^\alpha) \in M_i$ such that $(x^\alpha, y^\alpha) \to (x, y)$. Let $y^\alpha = (y_i^\alpha)_{i\in I}$ and $y = (y_i)_{i\in I}$. One has $x_i^\alpha \to x_i$, $y_i^\alpha \to y_i$, $y_i^\alpha \in T_i(x^\alpha)$, $g_i(x_i^\alpha, y^\alpha) \ge 0$, and $f_i(x^\alpha, y_i^\alpha, v_i) \ge 0$ for all $v_i \in T_i(x^\alpha)$. Let $v_i \in T_i(x)$. Since T_i is l.s.c., there exists a net $\{v_i^\alpha\}$ such that $v_i^\alpha \in T_i(x^\alpha)$ and $v_i^\alpha \to v_i$. Therefore $f_i(x^\alpha, y_i^\alpha, v_i^\alpha) \ge 0$. Since f_i and g_i are u.s.c. functions, $g_i(x_i, y) \ge 0$ and $f_i(x, y_i, v_i) \ge 0$. By assumption and Theorem 2.2 that T_i is closed. Hence $y_i \in T_i(x)$. This shows that $(x, y) \in M_i$ and M_i is a closed set for each $i \in I$. Since $M_i \subseteq X \times T_i(X)$ and $X \times T_i(X)$ is compact. M_i is a compact set for each $i \in I$. Let $M = \cap_{i\in I} M_i$, then M is a compact set. By Theorem 3.1 that $M \ne \emptyset$. Since $h\colon X \times Y \to \mathbb{R}$ is l.s.c. on M and M is a compact subset of $X \times Y$. Therefore, there exists $(\bar{x}, \bar{y}) \in M$ such that $h(\bar{x}, \bar{y}) = \min h(M)$. This shows that there exists a solution of problem (1).

Remark Theorem 3.2 is different from any results in [3, 8–11].

For the special cases of the Theorem 3.2, we have the following existence theorem of bilevel problem.

Corollary 3.1 *Let $I, X_i, X, Y_i, E_i, V_i, T_i, h_i$ and g_i be the same as in Theorem* 3.1. *Let f_i:$X \times Y_i \to \mathbb{R}$ be a continuous function such that for each $x \in X$, $v_i \to f_i(x, v_i)$ is quasi-convex for each fixed $x \in X$. Then there exists a solution of the problem:*
$\min\limits_{(x,y)} h(x, y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x_i, y) \geq 0$ and y_i is a solution of $Q_i(x)$:

$$\min_{v_i \in T_i(x)} f_i(x, v_i) \quad \text{for all } i \in I.$$

Proof Let $F_i(x, y_i, v_i)=f_i(x, v_i)-f_i(x, y_i)$. Then Corollary 3.1 follows from Theorem 3.2.

Remark In Corollary 3.1, if we assume further that $f_i(x, y_i) \geq 0$ for all $x \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$ and $g_i(x_i, y) \geq 0$. Then there exists a solution of the semi-infinite program:
$\min\limits_{(x,y)} h(x, y)$ such that $x = (x_i)_{i\in I} \in X$, $y = (y_i)_{i\in I} \in Y$, $y_i \in T_i(x)$, $g_i(x_i, y) \geq 0$ and $f_i(x, v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$.

Corollary 3.2 *Let $I, X_i, X, Y_i, E_i, V_i, T_i$ and h and g_i be the same as in Theorem* 3.1. *Let $H_i : X \to Y_i^*$ be a continuous function, $\eta_i : Y_i \times Y_i \to Y_i$ be an affine continuous function such that $\eta_i(y_i, y_i) = 0$ for all $y_i \in Y_i$, where Y_i^* is the dual space of Y_i and $\langle \cdot, \cdot \rangle$ will denote the dual pair between Y_i and Y_i^*. Let $p_i : Y_i \to \mathbb{R}$ be a continuous convex function. Then there exists a solution of the program:*

$$\min_{(x,y)} h(x, y) \text{ such that } x = (x_i)_{i\in I} \in X, y = (y_i)_{i\in I} \in Y, y_i \in T_i(x), g_i(x_i, y) \geq 0$$

and

$$\langle H_i(x), \eta_i(y_i, v_i) \rangle + p_i(v_i) - p_i(y_i) \geq 0 \quad \text{for all } v_i \in T_i(x)$$

and for all $i \in I$.

Proof Let $f_i : X \times Y_i \times Y_i \to \mathbb{R}$ be defined by

$$f_i(x, y_i, v_i) = \langle H_i(x), \eta_i(y_i, v_i) \rangle + p_i(v_i) - p_i(y_i).$$

Then $f_i : X \times Y_i \times Y_i \to \mathbb{R}$ is a continuous function and for each fixed $(x, y_i) \in X \times Y_i$, $v_i \to f_i(x, y_i, v_i)$ is quasiconvex. Indeed, if $v_i, v_i' \in Y_i$ and $\lambda \in [0, 1]$,

$$\begin{aligned}
&f_i(x, y_i, \lambda v_i + (1-\lambda)v_i') \\
&\quad = \langle H_i(x), \eta_i(y_i, \lambda v_i + (1-\lambda)v_i') \rangle + p_i(\lambda v_i + (1-\lambda)v_i') - p_i(y_i) \\
&\quad \leq \lambda \langle H_i(x), \eta_i(y_i, v_i) \rangle + (1-\lambda) \langle H_i(x), \eta_i(y_i, v_i') \rangle \\
&\quad\quad + \lambda p_i(v_i) + (1-\lambda) p_i(v_i') - p_i(y_i) \\
&\quad = \lambda f_i(x, y_i, v_i) + (1-\lambda) f_i(x, y_i, v_i') \\
&\quad \leq \max\{f_i(x, y_i, v_i), f_i(x, y_i, v_i')\}.
\end{aligned}$$

Hence $v_i \to f_i(x, y_i, v_i)$ is quasi-convex for each fixed $(x, y_i) \in X \times Y_i$. Similarly, $y_i \to f_i(x, y_i, v_i)$ is quasi-concave for each fixed $(x, v_i) \in X \times Y_i$.

$$f(x, y_i, y_i) = 0 \quad \text{for all } (x, y_i) \times X \times Y_i.$$

Then by Theorem 3.1, there exists a solution of the program:

$$\min_{(x,y)} h(x, y) \text{ such that } x = (x_i)_{i \in I} \in X, y = (y_i)_{i \in I}, y_i \in T_i(x), g_i(x_i, y) \geq 0$$

and

$$\langle H_i(x), \eta_i(y_i, v_i) \rangle + p_i(v_i) - p_i(y_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

Remark In Corollary 3.2, if I is a singleton, $p_i(v_i) = 0$ for all $v_i \in Y_i$. Then Corollary 3.2 will be reduced to the usual mathematical program with equilibrium constraint which was studied in [9].

Corollary 3.3 *Let I, X_i, E_i, Y_i, h, V_i and T_i be the same as in Theorem* 3.1. *Let f_i:$X \times Y_i \times Y_i \to \mathbb{R}$ be a function satisfying the following conditions:*

(1) *$f_i : X \times Y_i \times Y_i \to \mathbb{R}$ is an u.s.c. function;*
(2) *for each $(x, y_i) \in X \times Y_i$, $f_i(x, y_i, y_i) \geq 0$ and $v_i \to f_i(x, y_i, v_i)$ is quasi-convex; and*
(3) *for each $(x, v_i) \in X \times Y_i$, $y_i \to f_i(x, y_i, v_i)$ is quasi-concave.*

Then there exists a solution of the program:

$\min_{(x,y)} h(x, y)$ such that $x \in X, y = (y_i)_{i \in I} \in Y, y_i \in T_i(x)$ and $f_i(x, y_i, v_i) \geq 0$ for all $v_i \in T_i(x)$ and all $i \in I$.

Proof Letting $g_i = 0$ in Theorem 3.2.

Corollary 3.4 *In Corollary 3.3, if we assume further that $g_i : X \times Y_i \to \mathbb{R}$ is a function satisfying the following condition:*

(1) *$g_i : X \times Y_i \to \mathbb{R}$ is an u.s.c. function;*
(2) *for each $x \in X$, $y_i \to g_i(x, y_i)$ is quasi-concave;*
(3) *for each $x \in X$, there exists $y_i \in T_i(x)$ such that $g_i(x, y_i) \geq 0$.*

Then there exists a solution of the program:

$$\min_{(x,y)} h(x, y) \text{ such that } x \in X, y = (y_i)_{i \in I}, y_i \in T_i(x),$$

$$g_i(x, y_i) \geq 0 \text{ and } f_i(x, y_i, v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{for all } g_i(x, v_i) \geq 0 \text{ and for all } i \in I.$$

Proof Let $F_i(x) = \{y_i \in T_i(x) : g_i(x, y_i) \geq 0\}$. Then follow the same argument as in Theorem 3.1, and we can show that $F_i : X \multimap Y_i$ is an u.s.c, multivalued map with nonempty closed convex values. Then by Corollary 3.3, there exists a solution of the program:

$$\min_{(x,y)} h(x, y) \text{ such that } x \in X, y = (y_i)_{i \in I}, y_i \in F_i(x),$$

and $f_i(x, y_i, v_i) \geq 0 \quad$ for all $v_i \in F_i(x)$.

Therefore, the following program has a solution.

$$\min_{x,y} h(x,y) \text{ such that } x \in X, y = (y_i)_{i\in I}, y_i \in T_i(x),$$

$$g_i(x,y_i) \geq 0 \text{ and } f_i(x,y_i,v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{for all } g_i(x,v_i) \geq 0$$
$$\text{and for all } i \in I.$$

Remark (1) The function g_i defined in Theorem 3.1 and the function g_i defined in Corollary 3.4 are different.

If we let $g_i = 0$ in Corollary 3.4, then Corollary 3.4 reduces to Corollary 3.3. Therefore, Corollaries 3.3 and 3.4 are equivalent.

4 Mathematical programming with systems of equilibrium constraints of type II

In this section, we study the following mathematical programming with systems of equilibrium constraints of type II.

$$\min_{(x,y)} h(x,y), (x,y) \in H_i \quad \text{for all } i \in I, \tag{2}$$

where $H_i = \{(x,y) \in X \times K : y = (y_i)_{i\in I}, y_i \in T_i(x) \text{ and } f_i(x,y,v_i) \geq 0 \text{ for all } v_i \in T_i(x)\}$.

The following Lemmas are needed in this section.

Lemma 4.1 [7] *Let I be any index set and let X_i be a nonempty convex subset of a t.v.s. E_i, $X = \Pi_{i\in I} X_i$. For each $i \in I$, let $P_i, Q_i : X \multimap X_i$ be multivalued maps satisfying the following conditions:*

1. *for each $x \in X$, $coP_i(x) \subseteq Q_i(x)$;*
2. *for each $x = (x_i)_{i\in I} \in X$, $x_i \notin Q_i(x)$;*
3. *for each $y_i \in X_i$, $P_i^-(y_i)$ is open; and*
4. *there exists a nonempty compact subset K of X and a compact convex subset D_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exist $j \in I$ and $y_j \in X_j$ such that $x \in P_j^-(y_j)$.*

Then there exists $\bar{x} \in X$ such that $P_i(\bar{x}) = \emptyset$ for all $i \in I$.

Lemma 4.2 *Let X be a nonempty subset of a topological space E, I be any index set. For each $i \in I$, let Y_i be a nonempty convex subset of a Hausdorff t.v.s. V_i. Let $Y = \Pi_{i\in I} Y_i$, $f_i : X \times Y \times Y_i \to \mathbb{R}$ be a function and $T_i : X \multimap Y_i$ be a multivalued map with nonempty closed convex values satisfying the following conditions:*

1. *for each fixed $(x,v_i) \in X \times Y_i$, $y \to f_i(x,y,v_i)$ is u.s.c;*
2. *for each $(x,y) \in X \times Y$, $v_i \to f_i(x,y,v_i)$ is quasi-convex;*
3. *for each $x \in X$, $y = (y_i)_{i\in I} \in Y$, $f_i(x,y,y_i) \geq 0$;*
4. *there exists a compact subset K of Y and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for each $x \in X$, $y \in Y \setminus K$, there exist $j \in I$ and $v_j \in D \cap T_j(x)$ such that $f_j(x,y,v_j) < 0$.*

Then for each $x \in X$, there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in T_i(x)$ and

$$f_i(x,\bar{y},v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

Proof For each $i \in I$ and $x \in X$, let $A_i(x) : \Pi_{i\in I}T_i(x) \multimap T_i(x)$ be defined by

$$A_i(x)(y) = \{v_i \in T_i(x) : f_i(x, y, v_i) < 0\} \quad \text{for } y = (y_i)_{i\in I} \in \Pi_{i\in I}T_i(x).$$

By (2) and $T_i(x)$ is convex, $A_i(x)(y)$ is a convex set for each $x \in X$, $y \in Y$. By (3), $y_i \notin [A_i(x)(y)]$. By (1), for each $u_i \in T_i(x)$, $[A_i(x)]^-(u_i)$ is open in $T_i(x)$. By (4), for each $x \in X$ and each $y \in \Pi_{i\in I}T_i(x)\backslash K$ there exist $j \in I$ and a nonempty compact convex set $D_j \cap T_j(x)$ and $v_j \in D_j \cap T_j(x)$ such that $y \in [A_j(x)]^-(v_j)$.
Then it follows from Lemma 4.1 that there exists $\bar{y} \in Y$ such that $A_i(x)(\bar{y}) = \emptyset$ for all $i \in I$. That is for each $x \in X$, there exists $\bar{y} = (\bar{y}_i)_{i\in I} \in Y$ such that $\bar{y}_i \in T_i(x)$

$$f_i(x, \bar{y}, v_i) \geq 0 \quad \text{for all } i \in I \text{ and for all } v_i \in T_i(x).$$

If $T_i(x)$ is a nonempty compact convex subset of Y_i for each $x \in X$ and $i \in I$, then we the following Lemma.

Lemma 4.3 *Lemma 4.2 is true if condition (4) in Lemma 4.2 is replaced by (iv′) $T_i : X \multimap Y_i$ is a multivalued map with nonempty compact convex values.*

Proof Since $T_i(x)$ is compact for each $x \in X$ and $i \in I$, $\Pi_{i\in I}T_i(x)$ is compact for each $x \in X$ and condition (iv) of Lemma 4.1 is satisfied. Follow the same argument as in Lemma 4.2, we can prove Lemma 4.3

As a simple consequence of Lemma 4.3, we have the following theorems.

Theorem 4.1 *In Lemma 4.2, if we assume further that X is a nonempty compact subset of a Hausdorff topological space, $h{:}X \times Y \to \mathbb{R}$ is a l.s.c. function, $T_i : X \multimap Y_i$ is a continuous multivalued map with nonempty closed convex values and $f_i : X \times Y \times Y_i \to \mathbb{R}$ is an u.s.c. function. Then there exists a solution of the program (2).*

Proof By Lemma 4.2, for each $x \in X$, there exists $y = (y_i)_{i\in I} \in Y$ such that $y_i \in T_i(x)$ and $f_i(x, y, v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$. By assumption (4), $y \in K$. Follow the same argument as in Theorem 3.2, we can prove that H_i is a closed subset of $X \times K$ for each $i \in I$. Since $X \times K$ is compact, H_i is compact for all $i \in I$. Let $H = \cap_{i\in I}H_i$. Then H is a nonempty compact convex subset of $X \times K$. Since h is l.s.c., there exists $(\bar{x}, \bar{y}) \in H$ such that $h(\bar{x}, \bar{y}) = \min h(H)$. Therefore, there exists a minimizer of the problem (2).

Remark Theorem 4.1 is different from Theorem 5[8].

Theorem 4.2 *In Lemma* 4.3*, if we assume further that X is a nonempty compact subset of a Hausdorff topological space, $h{:}\ X \times Y \to \mathbb{R}$ is a l.s.c. function and $T_i{:}\ X \multimap Y_i$ is a continuous multivalued map with nonempty compact convex values. Then there exists a minimizer of the problem (2).*

Proof Let H_i and H be defined as in Theorem 4.1. By Lemma 4.3, $H \neq \emptyset$.

Since X is compact and $T_i : X \multimap Y_i$ is an u.s.c. multivalued map with nonempty compact values, it follows from Theorem 2.1 that $T_i(X)$ is compact. Following the same argument as in Theorem 4.1, we can show that H_i is closed. But $H_i \subseteq X \times \Pi_{i\in I}T_i(X)$ and $X \times \Pi_{i\in I}T_i(X)$ is compact. H_i is compact. Hence H is compact and the theorem follows from the fact that h is l.s.c.

Remark Theorem 4.1 is different from Theorem 5 [8]. For $\mathcal{F}_i \subset X \times Y_i$, by $\pi_X\mathcal{F}_i$ will denote the projection of $\mathcal{F}_i$ on X and by $\pi_{Y_i}\mathcal{F}_i$ will denote the projection of $\mathcal{F}_i$ on Y_i.

Theorem 4.3 *In Theorem* 4.1, *if we assume further that for each* $i \in I$, $g_i: X \times Y_i \to \mathbb{R}$ *is an u.s.c. quasi-concave function. Suppose that* $\mathcal{F}_i = \{(x, y_i) \in X \times Y_i : g_i(x, y_i) \geq 0\}$ *is nonempty,* $B_i = \pi_{Y_i}\mathcal{F}_i$, $A_i = \pi_X \mathcal{F}_i$, $A = \cap_{i\in I} A_i \neq \emptyset$, $T_i|_A : A \to B_i$, *and* $GrT_i|_A \subseteq \mathcal{F}_i$ *for each* $i \in I$. *Then there exists a solution of the program:*

$$\min_{(x,y)} h(x,y) \text{ such that } x \in X, \quad y = (y_i)_{i\in I}, y_i \in T_i(x),$$
$$g_i(x, y_i) \geq 0 \text{ and } f_i(x, y, v_i) \geq 0 \quad \text{for all } v_i \in T_i(x)$$

and for all $i \in I$.

Proof It is easy to see that $\mathcal{F}_i$ is a nonempty closed convex subset of $X \times Y_i$ for each $i \in I$. Therefore A_i is a nonempty closed convex subset of X and B_i is a nonempty closed convex of Y_i. Since X is compact, A is compact. By Theorem 3.3 there exist $x \in A$, $y = (y_i)_{i\in I}$, $y_i \in T_i(x)$, such that $f_i(x, y, v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$. Since $(x, y_i) \in GrT_i|_A \subseteq \mathcal{F}_i$, $g_i(x, y_i) \geq 0$ for all $i \in I$.

The following Lemma slightly generalizes Theorem 5(a) [8]. Although the proof is essentially the same, we give its proof for the sake of completeness.

Remark Theorem 4.3 is different from Theorem 6 [8].

Lemma 4.4 *Let* I *be any index set. For each* $i \in I$, *let* $f_i : X \times Y \times Y_i \to \mathbb{R}$ *be a quasi-concave function, and* $T_i : X \multimap Y_i$ *be a convex and concave multivalued map. Let*

$$H_i = \{(x, y) \in X \times Y : y = (y_i)_{i\in I}, y_i \in T_i(x) \text{ and } f_i(x, y, v_i) \geq 0 \text{ for all } v_i \in T_i(x)\}.$$

Then H_i *is a convex set for all* $i \in I$.

Proof Let (x, y) and $(x', y') \in H_i$ and $\lambda \in [0, 1]$. Then $x, x' \in X$, $y = (y_i)_{i\in I} \in Y$, $y' = (y_i')_{i\in I} \in Y$, $y_i \in T_i(x)$, $y_i' \in T_i(x')$, $f_i(x, y, v_i) \geq 0$ for all $v_i \in T_i(x)$ and $f_i(x', y', v_i') \geq 0$ for all $v_i' \in T_i(x')$. We have $(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y') \in X \times Y$. Since T_i is concave, $\lambda y_i + (1-\lambda)y_i' \in T_i(\lambda x + (1-\lambda)x')$. Let $u_i \in T_i(\lambda x + (1-\lambda)x')$. Since T_i is convex, there exist $v_i \in T_i(x)$, $v_i' \in T_i(x')$ such that $u_i = \lambda v_i + (1-\lambda)v_i'$. By quasi-convexity of f_i, either $0 \leq f_i(x, y, v_i) \leq f_i(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y', \lambda v_i + (1-\lambda)v_i')$ or

$$0 \leq f_i(x', y', v_i') \leq f_i(\lambda x + (1-\lambda)x', \qquad \lambda y + (1-\lambda)y', \lambda v_i + (1-\lambda)v_i').$$

In any case, $f_i(\lambda x + (1-\lambda)x', \lambda y + (1-\lambda)y', u_i) \geq 0$ for any $u_i \in T_i(\lambda x + (1-\lambda)x')$. This shows that $\lambda(x, y) + (1-\lambda)(x', y') \in H_i$ and H_i is convex.

Theorem 4.4 *Under the assumptions of Lemma* 4.2 *or* 4.3. *Suppose the assumptions of Lemma 4.4 hold. Let* $H = \cap_{i\in I} H_i$ *and* H_i *be defined as in Lemma 4.4. Suppose further that*

(1) $h: X \times Y \to \mathbb{R}$ *is a l.s.c. and quasiconcave function;*
(2) *there exist a nonempty compact subset* K *of* H *and a nonempty compact convex subset* C *of* H *such that for each* $(x, y) \in H \setminus K$, *there exists* $(u, v) \in C$ *such that* $h(u, v) < h(x, y)$.

Then there exists a solution of the problem (2).

Proof By Lemmas 4.2 or 4.3, there exists $(x, y) \in X \times Y$, $y = (y_i)_{i\in I}$, $y_i \in T_i(x)$ and $f_i(x, y, v_i) \geq 0$ for all $v_i \in T_i(x)$ and for all $i \in I$. Therefore $(x, y) \in H_i$ for all $i \in I$. This shows that $H = \cap_{i\in I} H_i \neq \emptyset$. By Lemma 4.4, H_i is convex for all $i \in I$, therefore

H is convex. Next we prove that the problem (2) has a solution. Let $P:H \to H$ be defined by

$$P(x,y) = \{(u,v) \in H : h(u,v) < h(x,y)\}.$$

Since h is quasi-convex, $P(x,y)$ is convex for each $(x,y) \in H$. Since h is l.s.c. $P^{-}(u,v) = \{(x,y) \in H : h(u,v) < h(x,y)\}$ is open in H. By (2), for each $(x,y) \in H\backslash K$, there exists $(u,v) \in C$ such that $(x,y) \in P^{-}(u,v)$. For each $(x,y) \in H$, $(x,y) \notin P(x,y)$. Then by Lemma 4.1 that there exists $(\bar{x},\bar{y}) \in H$ such that $P(\bar{x},\bar{y}) = \emptyset$. That is $h(u,v) \geq h(\bar{x},\bar{y})$ for all $(u,v) \in H$. This shows that the problem (2) has a solution.

Remark Theorem 4.4 improves Theorem 6 [8]. But because I may not be singleton here, in the proof we use maximal element theorem for a family of multivalued maps instead of KKM theorem which has been used to prove Theorem 6 of [8]. KKM theorem can not be applied to prove this theorem.

Applying Theorem 4.4 and following the same argument as in Theorem 4.3, we have the following theorem.

Theorem 4.5 *In Theorem* 4.4, *if we assume further that for each $i \in I$, $g_i: X \times Y_i \to \mathbb{R}$ is a quasi-concave function. Suppose that $\mathcal{F}_i = \{(x,y_i) \in X \times Y_i : g_i(x,y_i) \geq 0\}$ is nonempty, $B_i = \pi_Y \mathcal{F}_i$, $A_i = \pi_X \mathcal{F}_i$, $A = \cap_{i\in I} A_i \neq \emptyset$, $T_i|_A : A \multimap B_i$ and $GrT_i|_A \subseteq \mathcal{F}_i$ for each $i \in I$. Then there exists a solution of the program:*

$$\min_{(x,y)} h(x,y) \text{ such that } x \in X, y = (y_i)_{i\in I}, \quad y_i \in T_i(x),$$
$$g_i(x,y_i) \geq 0 \text{ and } f_i(x,y,v_i) \geq 0 \quad \text{for all } v_i \in T_i(x) \quad \text{and for all } i \in I.$$

As applications of Theorem 4.5, we establish the existence theorem of mathematical program with Nash equilibrium constraints.

Theorem 4.6 *Let $X, E, Y_i, V_i, Y, g_i, A_i, B_i, A, \mathcal{F}_i$ and T_i be the same as in Theorem* 4.5. *Let $\varphi_i : Y \times Y_i \to \mathbb{R}$ be a continuous function. Suppose that*

(1) *for each $y \in Y$, $v_i \to \varphi_i(y,v_i)$ is quasi-convex.*
(2) *there exist a compact subset K of Y and a nonempty compact convex subset D_i of Y_i for each $i \in I$ such that for each $x \in X$, $y \in Y\backslash K$ there exist $j \in I$ and $v_j \in D_j \cap T_j(x)$ such that $\varphi_j(y,v_j) < \varphi_j(y,y_j)$. Then there exists a minimizer to the program:*

$$\min_{(x,y)} h(x,y) \text{ such that } x \in X, \quad y = (y_i)_{i\in I}, y_i \in T_i(x),$$
$$g_i(x,y_i) \geq 0 \text{ and } \varphi_i(y,v_i) \geq \varphi(y,y_i) \quad \text{for all } v_i \in T_i(x) \text{ and all } i \in I.$$

Proof Let $f_i(x,y,v_i) = \varphi_i(y,v_i) - \varphi_i(y,y_i)$. Then Theorem 4.6 follows from Theorem 4.5.

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