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Some existence results for solutions of generalized vector quasi-equilibrium problems

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Abstract In this paper, we consider more general forms of generalized vector quasi-equilibrium problems for multivalued maps which include many known vector quasi-equilibrium problems and generalized vector quasi-variational inequality problems as special cases. We establish some existence results for solutions of these problems under pseudomonotonicity and u-hemicontinuity/ℓ-hemicontinuity assumptions.

Keywords Generalized vector quasi-equilibrium problems · u-Hemicontinuous · ℓ-Hemicontinuous · Quasi-equilibrium problem · Mixed-variational inequalities · Upper (lower) semicontinuity · Pseudomonotone maps

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1 Introduction and formulations

The vector variational inequalities, that is, variational inequalities for vector-valued functions, were introduced by Giannessi (1980) in finite dimensional spaces with

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further applications. Inspired by this pioneer work of Giannessi, the equilibrium problem Bianch and Schaible (1996), Blum and Oettli (1994) and Flores-Bazan (2000) which contains variational inequalities, optimization problems, saddle point problems, fixed point problem etc, has been extended and generalized for vector-valued functions in many different directions by many researchers, see for example Ansari and Yao (2003), Bianch et al. (1997), Hadijisavvas and Schaible (1998), Flores-Bazan and Flores-Bazan (2003), Giannessi (2000) and Konnov and Yao (1999) and references therein. The equilibrium problem with constrained is called *quasi-equilibrium problem*. The equilibrium problem and quasi-equilibrium problem for vector-valued functions are known as *vector equilibrium problem* and *vector quasi-equilibrium problem*, respectively. In the recent past vector (quasi-) equilibrium problems are used as tools to study vector (quasi-) optimization problem and vector (quasi-) saddle point problems, see for example Ansari and Flores-Bazán (2003), Ansari and Yao (2003), Lee (2000) and references therein.

In this paper, we consider more general vector quasi-equilibrium problems for multivalued functions which contain many known vector quasi-equilibrium problems and vector quasi-variational inequality problems as special cases. Several kinds of pseudomonotonicities and ℓ-hemicontinuities are introduced. We study the existence of a solution of these problems under some pseudomonotonicity and u-hemicontinuity/ℓ-hemicontinuity assumptions.

Throughout the paper, unless otherwise specified, we use the following notations and assumptions.

For a subset A of a topological vector space, we denote by co$A$, int A, $\overline{A}$ and 2^A the convex hull of A, the interior of A, the closure of A and the family of all subsets of A, respectively. Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be real topological vector spaces, X a nonempty convex subset of $\mathcal{X}$ and D a nonempty subset of $\mathcal{Y}$. Let $C : X \to 2^{\mathcal{Z}}$ be a multivalued map such that for each $x \in X$, $C(x)$ is a proper closed convex cone with apex at the origin. Let $S : X \to 2^X$, $T : X \to 2^D$ and $F : D \times X \times X \to 2^{\mathcal{Z}}$ be multivalued maps with nonempty values.

We consider the following types of *generalized vector quasi-equilibrium problems:*

$$\text{(GVQEP)(I)} \quad \begin{cases} \text{Find } \bar{x} \in X \text{ such that } \bar{x} \in S(\bar{x}) \text{ and} \\ \text{for each } y \in S(\bar{x}) \text{ there exists } t \in T(\bar{x}) \text{ satisfying} \\ F(t, \bar{x}, y) \nsubseteq -\text{int } C(\bar{x}). \end{cases}$$

$$\text{(GVQEP)(II)} \quad \begin{cases} \text{Find } \bar{x} \in X \text{ such that } \bar{x} \in S(\bar{x}) \text{ and} \\ F(t, \bar{x}, y) \cap \left(-\text{int } C(\bar{x})\right) = \emptyset \quad \text{for all } y \in S(\bar{x}) \text{ and } t \in T(\bar{x}). \end{cases}$$

$$\text{(GVQEP)(III)} \quad \begin{cases} \text{Find } \bar{x} \in X \text{ such that } \bar{x} \in S(\bar{x}) \text{ and} \\ F(t, \bar{x}, y) \subseteq C(\bar{x}) \quad \text{for all } y \in S(\bar{x}) \text{ and } t \in T(\bar{x}). \end{cases}$$

In (GVQEP)(I) and (GVQEP)(II), it is assumed that int $C(x) \neq \emptyset$ for each $x \in X$. We notice that very solution of (GVQEP)(II) is a solution of (GVQEP)(I).

Recently Fu and Wan (2002) considered and studied (GVQEP)(I) when $S(x) = X$ for all $x \in X$. It is worth to mention that the above three kinds of problems contain many problems studied in the literature as special cases; See for example

Ansari and Flores-Bazán (2003), Ansari and Yao (1999), Ansari and Yao (2003), Chen et al. (2001), Chiang et al. (2003), Khanh and Luu (2005), Lin and Park (1998), Lin and Yu (2001) and references therein.

2 Preliminaries

Definition 2.1 *Let $T : X \to 2^D$ be a multivalued map with nonempty values. A multivalued map $F : D \times X \times X \to 2^{\mathcal{Z}}$ is said to be u-*hemiconitnuous *(respectively, ℓ-*hemicontinuous*)* with respect to T *if for all $x, y \in X$ and $\alpha \in [0, 1]$, the multivalued map $\alpha \mapsto F(T(x_\alpha), x, y) = \bigcup_{t \in T(x_\alpha)} F(t, x, y)$ is upper semicontinuous (respectively, lower semicontinuous) at 0_+, where $x_\alpha = y + \alpha(x - y)$.*

Throughout the paper, all topological spaces are assumed to be Hausdorff.
The following results will be needed in the sequel.

Theorem 2.1 (Lin and Yu 2001) *Let $F : X \times Y \to 2^{\mathcal{Z}}$ and $S : X \to 2^Y$ be multivalued maps with nonempty values.*

(a) *If both S and F are lower semicontinuous, then $T : X \to 2^{\mathcal{Z}}$ defined by*

$$T(x) = \bigcup_{y \in S(x)} F(x, y) = F(x, S(x))$$

is lower semicontinuous on X.

(b) *If both F and S are upper semicontinuous with compact values, then T is an upper semicontinuous multivalued map with compact values.*

The following lemma will be needed to establish existence results for solutions of generalized mixed vector variational inequalities.

Lemma 2.1 (Ding and Tarafdar 2000) *Let $L(\mathcal{X}, \mathcal{Z})$ be the family of all continuous linear maps from $\mathcal{X}$ to $\mathcal{Z}$ such that it is equipped with the σ-topology. Then the bilinear mapping $\langle \cdot, \cdot \rangle : L(\mathcal{X}, \mathcal{Z}) \times \mathcal{X} \to \mathcal{Z}$ is continuous on $L(\mathcal{X}, \mathcal{Z}) \times \mathcal{X}$, where $\langle s, x \rangle$ denotes the evaluation of $s \in L(\mathcal{X}, \mathcal{Z})$ at $x \in \mathcal{X}$.*

The following result is a special case of Theorem 4.1 in Lin et al. (2003).

Theorem 2.2 *Let X be a nonempty convex subsets of a topological vector space $\mathcal{X}$. Let $S : X \to 2^X$ be a multivalued map such that*

(i) *for each $x \in X$, $x \notin \text{co}S(x)$;*
(ii) *for each $y \in X$, $S^{-1}(y) = \{x \in X : y \in S(x)\}$ is open in X;*
(iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset M of X such that for each $x \in X \backslash K$, there exists $\tilde{y} \in M$ such that $x \in S^{-1}(\tilde{y})$.*

Then there exists $\bar{x} \in X$ such that $S(\bar{x}) = \emptyset$.

3 Existence results for solutions of (GVQEP)

Let us recall the following definitions.

Definition 3.1 *For each fixed $x \in X$, a multivalued map $G : X \to 2^{\mathcal{Z}} \setminus \{\emptyset\}$ is called*

(i) C_x-convex *if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,*

$$G(\lambda x_1 + (1-\lambda)x_2) \subseteq \lambda G(x_1) + (1-\lambda)G(x_2) - C(x);$$

(ii) C_x-quasiconvex *if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$, either*

$$G(x_1) \subseteq G(\lambda x_1 + (1-\lambda)x_2) + C(x)$$

or

$$G(x_2) \subseteq G(\lambda x_1 + (1-\lambda)x_2) + C(x);$$

(iii) concave *if for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,*

$$\lambda G(x_1) + (1-\lambda)G(x_2) \subseteq G(\lambda x_1 + (1-\lambda)x_2).$$

Definition 3.2 *A multivalued map $F : D \times X \times X \to 2^{\mathcal{Z}} \setminus \{\emptyset\}$ is said to be*

(i) type-I C_x-pseudomontone with respect to T *if for all $x, y \in X$,*

$$F(u, x, y) \not\subseteq -\text{int}\, C(x) \quad \textit{for some } u \in T(x)$$
$$\textit{implies} \quad F(v, x, y) \not\subseteq -\text{int}\, C(x) \quad \textit{for some } v \in T(y);$$

(ii) type-II C_x-pseudomontone with respect to T *if for all $x, y \in X$,*

$$F(u, x, y) \cap \big(-\text{int}\, C(x)\big) = \emptyset \quad \textit{for some } u \in T(x)$$
$$\textit{implies} \;\; F(v, x, y) \cap \big(-\text{int}\, C(x)\big) = \emptyset \quad \textit{for all } v \in T(y);$$

(iii) type-III C_x-pseudomontone with respect to T *if for all $x, y \in X$,*

$$F(u, x, y) \subseteq C(x) \quad \textit{for some } u \in T(x)$$
$$\textit{implies} \quad F(v, x, y) \subseteq C(x) \quad \textit{for all } v \in T(y);$$

(iv) strong type-III maximal C_x-pseudomontone with respect to T *if it is type-III C_x-pseudomonotone w.r.t. T and for all $x, y \in X$, and each z in the line segment $(x, y]$ we have*

$$F(w, x, y) \subseteq C(x) \quad \textit{for some } w \in T(z)$$
$$\textit{implies} \quad F(u, x, y) \subseteq C(x) \quad \textit{for all } u \in T(x);$$

(v) strongly explicit $\delta(C_x)$-quasiconvex with respect to T *if for all $w \in Y$, $y_1, y_2 \in X$ and $\lambda \in (0, 1)$, we have either*

$$F(w, y_\lambda, y_1) \subseteq F(w, y_\lambda, y_\lambda) + C(y_1) \quad \text{or}$$
$$F(w, y_\lambda, y_2) \subseteq F(w, y_\lambda, y_\lambda) + C(y_2)$$

and in case $F(w, y_\lambda, y_1) - F(w, y_\lambda, y_2) \not\subseteq -C(y_1)$ for all $\lambda \in (0, 1)$ we have

$$F(w, y_\lambda, y_1) \subseteq F(w, y_\lambda, y_\lambda) + \text{int}\, C(y_1), \quad \text{where} \;\; y_\lambda = \lambda y_1 + (1-\lambda)y_2.$$

From now onward, we assume that the set $\mathscr{F} = \{x \in X : x \in S(x)\}$ is closed and the multivalued map $W : X \to 2^{\mathcal{Z}}$ is defined by $W(x) = \mathcal{Z}\backslash(-\text{int } C(x))$ for all $x \in X$.

First we present an existence result for a solution of (GVQEP)(I) under u-hemi-continuity assumption.

Theorem 3.1 *Let $T : X \to 2^D$ be a multivalued map with nonempty compact values, $S : X \to 2^X$ a multivalued map with nonempty convex values such that for all $y \in X$, $S^{-1}(y)$ is open in X, and the multivalued map $W : X \to 2^{\mathcal{Z}}$ be upper semicontinuous. Let $F : D \times X \times X \to 2^{\mathcal{Z}}$ be u-hemicontinuous and type-I C_x-pseudomonotone both w.r.t. T with nonempty values such that the following conditions hold:*

(i) *$F(t, x, x) \subseteq C(x)$ for all $x \in X$ and $t \in D$;*
(ii) *For each $(t, x) \in D \times X$, the multivalued map $y \mapsto F(t, x, y)$ is C_x-convex;*
(iii) *For each $y \in X$, the multivalued map $(t, x) \mapsto F(t, x, y)$ is upper semicontinuous with compact values;*
(iv) *There exist a nonempty compact set $K \subseteq X$ and a nonempty compact convex subset M of X such that for each $x \in X\backslash K$, there exists $\tilde{y} \in M \cap S(x)$ such that $F(s, x, \tilde{y}) \subseteq -\text{int } C(x)$ for all $s \in T(\tilde{y})$.*

Then (GVQEP)(I) has a solution.

Proof For each $x \in X$, define two multivalued maps $P, Q : X \to 2^X$ by

$$P(x) = \big\{y \in X : F(s, x, y) \subseteq -\text{int } C(x) \quad \text{for all } s \in T(y)\big\}$$

and

$$Q(x) = \big\{y \in X : F(t, x, y) \subseteq -\text{int } C(x) \quad \text{for all } t \in T(x)\big\}.$$

Since F is type-I pseudomonotone w.r.t. T, $P(x) \subseteq Q(x)$ for all $x \in X$. Also, for all $x \in X$, $Q(x)$ is convex.

Indeed, let $y_1, y_2 \in Q(x)$ and $\lambda \in [0, 1]$, then by (ii), for each fixed $t \in T(x)$,

$$\begin{aligned} F(t, x, \lambda y_1 + (1-\lambda)y_2) &\subseteq \lambda F(t, x, y_1) + (1-\lambda)F(t, x, y_2) - C(x) \\ &\subseteq -\text{int } C(x) - \text{int } C(x) - C(x) \subseteq -\text{int } C(x). \end{aligned}$$

Therefore $\lambda y_1 + (1-\lambda)y_2 \in Q(x)$ and hence $Q(x)$ is convex.

Since for each fixed $y \in X$, the multivalued map $(t, x) \mapsto F(t, x, y)$ is upper semicontinuous with compact values and $T(y)$ is compact for all $y \in X$, it follows from Theorem 2.1 or Theorem 1 in Flores-Bazan and Flores-Bazan (2003) that for each fixed $y \in X$, $x \mapsto F(T(y), x, y)$ is an upper semicontinuous multivalued map with compact values.

Now we show that for all $y \in X$, $P^{-1}(y) = \{x \in X : F(T(y), x, y) \subseteq -\text{int } C(x)\}$ is open in X.

Indeed, let $x \in \overline{X\backslash P^{-1}(y)}$, then there exists a net $\{x_\alpha\}_{\alpha\in\Lambda}$ in $X\backslash P^{-1}(y)$ such that $x_\alpha \to x \in X$. Therefore, $x_\alpha \in X$ and $F(T(y), x_\alpha, y) \not\subseteq -\text{int } C(x_\alpha)$, that is, $F(T(y), x_\alpha, y) \cap [\mathcal{Z}\backslash(-\text{int } C(x_\alpha))] \neq \emptyset$. Let $z_\alpha \in F(T(y), x_\alpha, y) \cap [\mathcal{Z}\backslash(-\text{int } C(x_\alpha))]$ and $L = \{x_\alpha : \alpha \in \Lambda\} \cup \{x\}$. Then L is compact. Since for each fixed $y \in X, x \mapsto F(T(y), x, y)$ is an upper semicontinuous multivalued map with compact values, it follows from Proposition 1.3 in Ansari and Yao (2003) that $F(T(y), L, y)$ is compact and $\{z_\alpha\}_{\alpha \in \Lambda}$ has a subnet $\{z_{\alpha_j}\}$ such that $z_{\alpha_j} \to z \in F(T(y), L, y)$. Since $W : X \to 2^{\mathcal{Z}}$ and $x \mapsto F(T(y), x, y)$ are upper semicontinuous with closed values, it follows from Theorem 2.1 that W and $x \mapsto F(T(y), x, y)$ are closed. Therefore, $z \in F(T(y), x, y) \cap W(x) \neq \emptyset$ and hence $F(T(y), x, y) \nsubseteq -\text{int}\, C(x)$. This implies that $x \notin P^{-1}(y)$ and thus $x \in X \backslash P^{-1}(y)$, for all $y \in X$. Therefore $X \backslash P^{-1}(y)$ is closed for all $y \in X$. Hence for all $y \in X$, $P^{-1}(y)$ is open in X.

We define

$$G(x) = \begin{cases} S(x) \cap P(x) & \text{if } x \in \mathscr{F} \\ S(x) & \text{if } x \notin \mathscr{F}. \end{cases}$$

By (i), $x \notin Q(x) = \text{co}Q(x)$ for all $x \in X$. Therefore $x \notin \text{co}P(x)$ and thus $x \notin \text{co}G(x)$ for all $x \in X$. It is easy to see that

$$G^{-1}(y) = [S^{-1}(y) \cap P^{-1}(y)] \cup [(X \backslash \mathscr{F}) \cap S^{-1}(y)].$$

Since $S^{-1}(y)$ and $P^{-1}(y)$ are open in X, we have $G^{-1}(y)$ is open in X for all $y \in X$. Condition (iv) implies for each $x \in X \backslash K$, there exists $\tilde{y} \in M$ such that $x \in G^{-1}(\tilde{y})$. It follows from Theorem 2.2 that there exists $\bar{x} \in X$ such that $G(\bar{x}) = \emptyset$. Since for all $x \in X$, $S(x)$ is nonempty, we have $S(\bar{x}) \cap P(\bar{x}) = \emptyset$ and $\bar{x} \in \mathscr{F}$. This shows that $\bar{x} \in S(\bar{x})$ and for each $y \in S(\bar{x})$, there exists $t \in T(y)$ such that

$$F(t, \bar{x}, y) \nsubseteq -\text{int}\, C(\bar{x}). \tag{1}$$

Finally, we show that for each $y \in S(\bar{x})$, there exists $t \in T(\bar{x})$ such that

$$F(t, \bar{x}, y) \nsubseteq -\text{int}\, C(\bar{x}). \tag{2}$$

Suppose to the contrary that there exists $\hat{y} \in S(\bar{x})$ such that

$$F(t, \bar{x}, \hat{y}) \subseteq -\text{int}\, C(\bar{x}) \quad \text{for all } t \in T(\bar{x}).$$

Since F is u-hemicontinuous w.r.t. T, there exists $\delta \in (0, 1)$ such that for all $\alpha \in (0, \delta)$, $x_\alpha = \bar{x} + \alpha(\hat{y} - \bar{x}) \in S(\bar{x})$ and

$$F(T(x_\alpha), \bar{x}, \hat{y}) \subseteq -\text{int}\, C(\bar{x}).$$

That is,

$$F(t, \bar{x}, \hat{y}) \subseteq -\text{int}\, C(\bar{x}) \quad \text{for all } t \in T(x_\alpha).$$

Since for each fixed $(t, x) \in D \times X$, $y \mapsto F(t, x, y)$ is C_x-convex and by (i), we have

$$\begin{aligned} F(t, \bar{x}, x_\alpha) &\subseteq \alpha F(t, \bar{x}, \hat{y})) + (1 - \alpha) F(t, \bar{x}, \bar{x}) - C(\bar{x}) \\ &\subseteq -\text{int}\, C(\bar{x}) + C(\bar{x}) - C(\bar{x}) \subseteq -\text{int}\, C(\bar{x}), \end{aligned}$$

for all $t \in T(x_\alpha)$, which is a contradiction of (1). This completes the proof. □

Remark 3.1 (a) If $S(x) = X$ for all $x \in X$, then Theorem 3.1 reduces to Theorem 3 in Fu and Wan (2002).

(b) For Different suitable choices of F in Theorem 3.1, we can easily derive the existence results for solutions of the problems considered in Ansari and Flores-Bazán (2003), Ansari and Yao (1999, 2003), Chiang et al. (2003), Fu and Wan (2002), Khanh and Luu (2005) and references therein.

Corollary 3.1 *Let $L(\mathcal{X}, \mathcal{Z})$ be equipped with the σ-topology. Let $T : X \to L(\mathcal{X}, \mathcal{Z})$ be a multivalued map with compact values, $S : X \to 2^X$ a multivalued map with nonempty convex values such that for all $y \in X$, $S^{-1}(y)$ is open in X, and the multivalued map $W : X \to 2^{\mathcal{Z}}$ be upper semicontinuous. Let $g : X \times X \to \mathcal{Z}$ be continuous function such that the following conditions hold:*

(i) $g(x, x) = 0$ *for all* $x \in X$;
(ii) *For all* $x, y \in X$, $\langle u, y - x\rangle + g(x, y) \notin -\text{int}\, C(x)$ *for some* $u \in T(x)$ *implies* $\langle v, y - x\rangle + g(x, y) \notin -\text{int}\, C(x)$ *for all* $v \in T(y)$;
(iii) *The multivalued map* $(t, x, y) \mapsto \langle t, y - x\rangle + g(x, y)$ *is u-hemicontinuous w.r.t. T, that is, for all $x, y \in X$ and $\alpha \in [0, 1]$, the multivalued map* $\alpha \mapsto \langle T(x_\alpha), y - x_\alpha\rangle + g(x_\alpha, y)$ *is upper semicontinuous, where* $x_\alpha = y + \alpha(x - y)$;
(iv) *For each* $(t, x) \in L(\mathcal{X}, \mathcal{Z}) \times X$, $y \mapsto \langle t, y - x\rangle + g(x, y)$ *is* C_x*-convex;*
(v) *There exist a compact set $K \subseteq X$ and a compact convex subset M of X such that for each $x \in X\backslash K$, there exists $\tilde{y} \in M \cap S(x)$ such that* $\langle s, \tilde{y} - x\rangle + g(x, \tilde{y}) \in -\text{int}\, C(x)$ *for all* $s \in T(\tilde{y})$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and for each $y \in S(\bar{x})$, there exists $t \in T(\bar{x})$ satisfying $\langle t, y - \bar{x}\rangle + g(\bar{x}, y) \notin -\text{int}\, C(\bar{x})$.

Proof Let $F(t, x, y) = \{\langle t, y - x\rangle + g(x, y)\}$. Since g is continuous, it follows from Lemma 2.1 that for each $y \in X$, $(t, x) \mapsto F(t, x, y)$ is an upper semicontinuous multivalued map with compact values, and the result follows from Theorem 3.1. □

Remark 3.2 From Theorem 3.1 we can easily derive some results on the existence of solutions of the problems considered in Fu and Wan (2002) and Khanh and Luu (2005).

Next we establish the existence result for a solution of (GVQEP)(II) under ℓ-hemicontinuity assumption.

Theorem 3.2 *Let $T : X \to 2^D$ be a multivalued map with nonempty values, $S : X \to 2^X$ a multivalued map with nonempty convex values such that for all $y \in X$, $S^{-1}(y)$ is open in X, and the multivalued map $W : X \to 2^{\mathcal{Z}}$ be upper semicontinuous. Let $F : D \times X \times X \to 2^{\mathcal{Z}}$ be ℓ-hemicontinuous and type-II C_x-pseudomonotone both w.r.t. T with nonempty values such that the following conditions hold:*

(i) $0 \in F(t, x, x) \subseteq C(x)$ *for all* $x \in X$ *and* $t \in D$;
(ii) *For each* $(t, x) \in D \times X$, *the multivalued map* $y \mapsto F(t, x, y)$ *is concave;*
(iii) *For each* $y \in X$, *the multivalued map* $(t, x) \mapsto F(t, x, y)$ *is lower semicontinuous;*
(iv) *There exist a compact set $K \subseteq X$ and a compact convex subset M of X such that for each $x \in X\backslash K$, there exists $\tilde{y} \in M \cap S(x)$ and $s \in T(\tilde{y})$ such that* $F(s, x, \tilde{y}) \cap (-\text{int}\, C(x)) \neq \emptyset$.

Then (GVQEP)(II) has a solution.

Proof For each $x \in X$, define two multivalued maps $P, Q : X \to 2^X$ by

$$P(x) = \{y \in X : F(s, x, y) \cap (-\text{int } C(x)) \neq \emptyset \text{ for some } s \in T(y)\}$$

and

$$Q(x) = \{y \in X : F(t, x, y) \cap (-\text{int } C(x)) \neq \emptyset \text{ for all } t \in T(x)\}.$$

Since F is type-II C_x-pseudomonotone w.r.t. T, $P(x) \subseteq Q(x)$ for all $x \in X$. Also, for all $x \in X$, $Q(x)$ is convex.

Indeed, if $y_1, y_2 \in Q(x)$ and $\lambda \in [0, 1]$, then for each $t \in T(x)$, we have

$$F(t, x, y_i) \cap (-\text{int } C(x)) \neq \emptyset \quad \text{for } i = 1, 2.$$

Let $u_i \in F(t, x, y_i) \cap (-\text{int } C(x))$, $i = 1, 2$. By (ii), for each fixed $(t, x) \in D \times X$,

$$\lambda u_1 + (1 - \lambda)u_2 \in \lambda F(t, x, y_1) + (1 - \lambda)F(t, x, y_2) \subseteq F(t, x, \lambda y_1 + (1 - \lambda)y_2).$$

Since $C(x)$ is convex for all $x \in X$, $\lambda u_1 + (1 - \lambda)u_2 \in (-\text{int } C(x))$ and therefore

$$F(t, x, \lambda y_1 + (1 - \lambda)y_2) \cap (-\text{int } C(x)) \neq \emptyset.$$

Hence $\lambda y_1 + (1 - \lambda)y_2 \in Q(x)$ and thus $Q(x)$ is convex.

By (i), $x \notin Q(x) = \text{co}Q(x)$ for all $x \in X$. Therefore $x \notin \text{co}P(x)$ for all $x \in X$. From (iii) and Theorem 2.1, we have that for each $y \in X$, $F(T(y), x, y)$ is lower semicontinuous.

Now we will show that for all $y \in X$, $P^{-1}(y) = \{x \in X : F(s, x, y) \cap (-\text{int } C(x)) \neq \emptyset \text{ for some } s \in T(y)\}$ is open in X.

Indeed, let $x \in \overline{X \setminus P^{-1}(y)}$, then there exists a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $X \setminus P^{-1}(y)$ such that $x_\alpha \to x \in X$. Therefore $x_\alpha \in X$ and $F(s, x_\alpha, y) \cap (-\text{int } C(x_\alpha)) = \emptyset$ for all $s \in T(y)$, that is, $F(T(y), x_\alpha, y) \cap (-\text{int } C(x_\alpha)) = \emptyset$ which implies that $F(T(y), x_\alpha, y) \subseteq \mathcal{Z} \setminus (-\text{int } C(x_\alpha)) = W(x_\alpha)$. Let $z \in F(T(y), x, y)$, then by the lower semicontinuity of F in the second argument, there exists a net $\{z_\alpha\}$ such that $z_\alpha \in F(T(y), x_\alpha, y)$ for all α and $z_\alpha \to z$. Therefore $z_\alpha \in W(x_\alpha)$. Since W is an upper semicontinuous multivalued map with closed values, it is closed. Therefore $z \in W(x)$ and $F(T(y), x, y) \subseteq W(x) = \mathcal{Z} \setminus (-\text{int } C(x))$, that is, $F(T(y), x, y) \cap (-\text{int } C(x)) = \emptyset$ and thus $x \in X \setminus P^{-1}(y)$. This shows that $X \setminus P^{-1}(y)$ is closed for all $y \in X$. Hence $P^{-1}(y)$ is open for all $y \in X$.

We define

$$G(x) = \begin{cases} S(x) \cap P(x) & \text{if } x \in \mathscr{F} \\ S(x) & \text{if } x \notin \mathscr{F}. \end{cases}$$

Since for all $x \in X$, $x \notin \text{co}P(x)$ we have that $x \notin \text{co}G(x)$ for all $x \in X$. It is easy to see that

$$G^{-1}(y) = [S^{-1}(y) \cap P^{-1}(y)] \cup [(X \setminus \mathscr{F}) \cap S^{-1}(y)].$$

Since $S^{-1}(y)$ and $P^{-1}(y)$ are open in X, we have that $G^{-1}(y)$ is open in X for all $y \in X$. Condition (iv) implies for each $x \in X \setminus K$, there exists $\tilde{y} \in M$ such that $x \in G^{-1}(\tilde{y})$. It follows from Theorem 2.2 that there exists $\bar{x} \in X$ such that

$G(\bar{x}) = \emptyset$. Since for all $x \in X$, $S(x)$ is nonempty, we have $S(\bar{x}) \cap P(\bar{x}) = \emptyset$ and $\bar{x} \in \mathscr{F}$. This shows that $\bar{x} \in S(\bar{x})$ and

$$F(T(y), \bar{x}, y) \cap (-\text{int } C(\bar{x})) = \emptyset \quad \text{for all } y \in S(\bar{x}).$$

Therefore,

$$F(s, \bar{x}, y) \cap (-\text{int } C(\bar{x})) = \emptyset \quad \text{for all } y \in S(\bar{x}) \text{ and for all } s \in T(y). \tag{3}$$

We want to show that

$$F(t, \bar{x}, y) \cap (-\text{int } C(\bar{x})) = \emptyset \quad \text{for all } y \in S(\bar{x}) \text{ and all } t \in T(\bar{x}). \tag{4}$$

If this is not true, then there exists $\hat{y} \in S(\bar{x})$ and $\hat{t} \in T(\bar{x})$ such that

$$F(\hat{t}, \bar{x}, \hat{y}) \cap (-\text{int } C(\bar{x})) \neq \emptyset.$$

Then $F(T(\bar{x}), \bar{x}, \hat{y}) \cap (-\text{int } C(\bar{x})) \neq \emptyset$. Let $x_\alpha = \alpha\hat{y} + (1-\alpha)\bar{x}, \alpha \in [0, 1]$. Then $x_\alpha \in S(\bar{x})$. Since F is ℓ-hemicontinuous w.r.t. T there exists $\delta > 0$ such that

$$F(T(x_\alpha), \bar{x}, \hat{y}) \cap (-\text{int } C(\bar{x})) \neq \emptyset \quad \text{for } 0 < \alpha < \delta.$$

There exists $t_1 \in T(x_\alpha)$ such that $F(t_1, \bar{x}, \hat{y}) \cap (-\text{int } C(\bar{x})) \neq \emptyset$. Let $u \in F(t_1, \bar{x}, \hat{y}) \cap (-\text{int } C(\bar{x}))$. Since $0 \in F(t_1, \bar{x}, \bar{x})$ and $y \mapsto F(t_1, \bar{x}, y)$ is concave, we have that for $\alpha \in [0, 1]$,

$$\begin{aligned} \alpha u = \alpha u + (1-\alpha)0 &\in \alpha F(t_1, \bar{x}, \hat{y}) + (1-\alpha)F(t_1, \bar{x}, \bar{x}) \\ &\subseteq F(t_1, \bar{x}, \alpha\hat{y} + (1-\alpha)\bar{x}) = F(t_1, \bar{x}, x_\alpha). \end{aligned}$$

Since $u \in (-\text{int } C(\bar{x}))$, we have $\alpha u \in (-\text{int } C(\bar{x}))$. Hence

$$\alpha u \in F(t_1, \bar{x}, x_\alpha) \cap (-\text{int } C(\bar{x})) \neq \emptyset \quad \text{for } \alpha \in [0, 1] \text{ and some } t_1 \in T(x_\alpha)$$

This contradicts with (3). Therefore

$$F(t, \bar{x}, y) \cap (-\text{int } C(\bar{x})) = \emptyset \quad \text{for all } t \in T(\bar{x}) \text{ and all } y \in S(\bar{x}).$$

This completes the proof. □

Remark 3.3 (a) If F is not necessarily ℓ-hemicontinuous w.r.t. T in Theorem 3.2, then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and $F(s, \bar{x}, y) \cap (-\text{int } C(\bar{x})) = \emptyset$ for all $y \in S(\bar{x})$ and for all $s \in T(y)$.

(b) Best of our knowledge, no existence result for solutions of generalized vector quasi-equilibrium problems is appeared in the literature. Therefore Theorem 3.2 is new in this area.

Definition 3.3 *Let $x \in X$ be any fixed element. A map $h : X \to \mathcal{Z}$ is said to be* concave *if for any $y_1, y_2 \in X$ and $\lambda \in [0, 1]$,*

$$\lambda h(y_1) + (1-\lambda)h(y_2) \in h(\lambda y_1 + (1-\lambda)y_2) + C(x).$$

The following result can be easily derived from Theorem 3.2.

Corollary 3.2 *Let $L(\mathcal{X}, \mathcal{Z})$ be equipped with the σ-topology. Let $T : X \to L(\mathcal{X}, \mathcal{Z})$ be a multivalued map with nonempty values, $S : X \to 2^X$ a multivalued map with nonempty convex values such that for all $y \in X$, $S^{-1}(y)$ is open in X, and the multivalued map $W : X \to 2^{\mathcal{Z}}$ be upper semicontinuous. Let $g : X \times X \to \mathcal{Z}$ be continuous function such that the following conditions hold.*

(i) *$g(x, x) = 0$ for all $x \in X$;*
(ii) *For all $x, y \in X$, $\big(\langle u, y-x\rangle + g(x, y)\big) \cap \big(-\operatorname{int} C(x)\big) \neq \emptyset$ for some $u \in T(x)$ implies $\big(\langle v, y - x\rangle + g(x, y)\big) \cap \big(-\operatorname{int} C(x)\big) \neq \emptyset$ for all $v \in T(y)$;*
(iii) *For each $(t, x) \in D \times X$, the map $y \mapsto \langle t, y - x\rangle + g(x, y)$ is concave;*
(iv) *There exist a compact set $K \subseteq X$ and a compact convex subset M of X such that for each $x \in X \backslash K$, there exist $\tilde{y} \in M \cap S(x)$ and $s \in T(\tilde{y})$ such that $\big(\langle s, \tilde{y} - x\rangle + g(x, \tilde{y})\big) \cap \big(-\operatorname{int} C(x)\big) \neq \emptyset$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and for all $y \in S(\bar{x})$ and all $s \in T(y)$, $\big(\langle s, y - \bar{x}\rangle + g(\bar{x}, y)\big) \cap \big(-\operatorname{int} C(\bar{x})\big) = \emptyset$.

Remark 3.4 In addition to the assumption of Corollary 3.2, if

(iv) the multivalued map $(t, x, y) \mapsto \langle t, y - x\rangle + g(x, y)$ is ℓ-hemicontinuous w.r.t. T, that is, for all $x, y \in X$ and $\alpha \in [0, 1]$, the multivalued map $\alpha \mapsto \langle T(x_\alpha), y - x_\alpha\rangle + g(x_\alpha, y)$ is lower semicontinuous, where $x_\alpha = y + \alpha(x - y)$.

Then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and

$$\big(\langle t, y - \bar{x}\rangle + g(\bar{x}, y)\big) \cap \big(-\operatorname{int} C(\bar{x})\big) = \emptyset \quad \text{for all } y \in S(\bar{x}) \text{ and all } t \in T(\bar{x}).$$

Proposition 3.1 *If F is strong type-III maximal C_x-pseudomonotone w.r.t. T, then (GVQEP)(III) is equivalent to the following problem:*

$$\text{(DGVQEP)(IV)} \quad \begin{cases} \textit{Find } \bar{x} \in X \textit{ such that } \bar{x} \in S(\bar{x}) \textit{ and} \\ F(t, \bar{x}, y) \subseteq C(\bar{x}) \quad \textit{for all } y \in S(\bar{x}) \textit{ and } t \in T(\bar{x}). \end{cases}$$

Proof By strong type-III maximal C_x-pseudomonotone w.r.t. T of F, we have (GVQEP)(III) implies (DGVQEP)(IV).

If $\bar{x}$ is a solution of (DGVQEP)(IV), then $F(s, \bar{x}, y) \subseteq C(\bar{x})$ for all $y \in S(\bar{x})$ and for all $t \in T(y)$. Since $S(\bar{x})$ is convex, we have the line segment $(\bar{x}, y) \subseteq S(\bar{x})$. Therefore for each z in the line segment $(x, y]$ and for all $s \in T(z)$, we have $F(s, \bar{x}, z) \subseteq C(\bar{x})$. Since F is strong type-III maximal C_x-pseudomonotone w.r.t. T, we have $F(s, \bar{x}, y) \subseteq C(\bar{x})$ for all $y \in S(\bar{x})$ and $t \in T(\bar{x})$. Hence $\bar{x}$ is a solution of (GVQEP)(III). □

Following the arguments of the proof of Theorem 3.1 in Lin (2005), we have the following result.

Proposition 3.2 *Let D be a nonempty convex subset of Y. Let $C : X \to 2^{\mathcal{Z}}$ be a multivalued map such that for each $x \in X$, $C(x)$ is a proper closed convex cone with* int $C(x) \neq \emptyset$. *Let $F : D \times X \times X \to 2^{\mathcal{Z}}$ and $T : X \to 2^D$ be multivalued maps with nonempty values. Assume that the following conditions hold:*

(i) *For all $t \in D$ and $x, y \in X$, $F(t, y, y) \subseteq C(x)$.*
(ii) *F is strongly explicit $\delta(C_x)$-quasiconvex and type-III C_x-pseudomonotone.*

(iii) *For each* $y \in X$ *and* $s \in D$*, the multivalued map* $x \mapsto F(s, x, y)$ *is lower semicontinuous.*
(iv) *For all* $s \in D$ *and* $x, y \in X$, $F(s, x, y) = -F(s, y, x)$.

Then F is strong type-III maximal C_x-pseudomonotone.

Proof Assume that $x, y \in X$, z in the line segment $(x, y]$ and $q \in T(z)$, $F(q, x, z) \subseteq C(x)$. Then we will have to show that $F(t, x, y) \subseteq C(x)$ for all $t \in T(x)$. Suppose that $F(t_0, x, y) \nsubseteq C(x)$ for some $t_0 \in T(x)$. Then $F(t_0, x, y) \cap [\mathcal{Z} \setminus C(x)] \neq \emptyset$. Since for each $x \in X$, $C(x)$ is closed and by (iii) we have for each $x, y \in X$, there exists $\alpha \in (0, 1)$ such that

$$F(t_0, x_\alpha, y) \cap [\mathcal{Z} \setminus C(x)] \neq \emptyset, \quad \text{where } x_\alpha = \alpha y + (1 - \alpha)x. \tag{5}$$

By (ii), either

$$F(t_0, x_\alpha, y) \subseteq F(t_0, x_\alpha, x_\alpha) + C(x) \subseteq C(x) + C(x) \subseteq C(x)$$

or

$$F(t_0, x_\alpha, x) \subseteq F(t_0, x_\alpha, x_\alpha) + C(x) \subseteq C(x) + C(x) \subseteq C(x)$$

The first relation contradicts with (5). Thus we have

$$F(t_0, x_\alpha, x) \subseteq C(x). \tag{6}$$

Hence $F(t_0, x_\alpha, x) - F(t_0, x_\alpha, y) \nsubseteq -C(x)$ follows from (5) and (6). Therefore

$$F(t_0, x_\alpha, x) \subseteq F(t_0, x_\alpha, x_\alpha) + \text{int } C(x) \subseteq C(x) + \text{int } C(x) \subseteq C(x). \tag{7}$$

Since $C(x)$ is proper, $-C(x) \cap (\text{int } C(x)) = \emptyset$ for all $x \in X$. By (7), $F(t_0, x_\alpha, x) \nsubseteq -\text{int } C(x)$. Hence $F(t_0, x_\alpha, x) \nsubseteq -C(x)$. This contradicts with $F(q, x, z) \subseteq C(x)$ for each z in the line segment $(x, y]$ and all $q \in T(z)$. □

Finally we present a result on the existence of a solution of (GVQEP)(III).

Theorem 3.3 *Let* $T : X \to 2^D$ *be a multivalued map with nonempty compact values,* $S : X \to 2^X$ *a multivalued map with nonempty convex values such that for all* $y \in X$, $S^{-1}(y)$ *is open in* X*, and the multivalued map* $C : X \to 2^{\mathcal{Z}}$ *be upper semicontinuous. Let* $F : D \times X \times X \to 2^{\mathcal{Z}}$ *be u-hemicontinuous and type-III* C_x*-pseudomonotone both w.r.t.* T *with nonempty values such that the following conditions hold.*

(i) $F(t, x, x) \subseteq C(x)$ *for all* $x \in X$ *and* $t \in D$;
(ii) *For each* $(t, x) \in D \times X$*, the multivalued map* $y \mapsto F(t, x, y)$ *is* C_x*-quasi-convex;*
(iii) *For each* $y \in X$*, the multivalued map* $(t, x) \mapsto F(t, x, y)$ *is lower semicontinuous with compact values;*
(iv) F *is strong type-III maximal* C_x*-pseudomonotone w.r.t.* T;
(v) *There exist a nonempty compact set* $K \subseteq X$ *and a nonempty compact convex subset* M *of* X *such that for each* $x \in X \setminus K$*, there exists* $\tilde{y} \in M \cap S(x)$ *such that* $F(s, x, \tilde{y}) \nsubseteq C(x)$ *for all* $s \in T(\tilde{y})$.

Then (GVQEP)(III) has a solution.

Proof For each $x \in X$, define two multivalued maps $P, Q : X \to 2^X$ by

$$P(x) = \{y \in X : F(s, x, y) \not\subseteq C(x) \text{ for some } s \in T(y)\}$$

and

$$Q(x) = \{y \in X : F(t, x, y) \not\subseteq C(x) \text{ for all } t \in T(x)\}.$$

Since F is type-III C_x-pseudomonotone w.r.t. T, we have $P(x) \subseteq Q(x)$ for all $x \in X$. Also, for all $x \in X$, $Q(x)$ is convex.

Indeed, let $y_1, y_2 \in Q(x)$ and $\lambda \in [0, 1]$, then $F(t, x, y_1) \not\subseteq C(x)$ and $F(t, x, y_2) \not\subseteq C(x)$ for all $t \in T(x)$. Suppose to the contrary that there exist $\lambda_0 \in [0, 1]$ and $t_0 \in T(x)$ such that $F(t_0, x, \lambda_0 y_1 + (1 - \lambda_0)y_2) \subseteq C(x)$. By (ii), either

$$F(t_0, x, y_1) \subseteq F(t_0, x, \lambda_0 y_1 + (1 - \lambda_0)y_2) + C(x) \subseteq C(x) + C(x) \subseteq C(x)$$

or

$$F(t_0, x, y_2) \subseteq F(t_0, x, \lambda_0 y_1 + (1 - \lambda_0)y_2) + C(x) \subseteq C(x).$$

This leads to a contradiction with $F(t, x, y_i) \not\subseteq C(x)$ for all $t \in T(x)$ and $i = 1, 2$. Therefore, $F(t, x, \lambda y_1 + (1-\lambda)y_2) \not\subseteq C(x)$ for all $t \in T(x)$ and for all $\lambda \in [0, 1]$. Hence $Q(x)$ is convex for all $x \in X$.

By (iii) and Theorem 2.1 that for each fixed $y \in X$, the multivalued map $(t, x) \mapsto F(T(y), x, y)$ is lower semicontinuous.

Now we show that for all $y \in X$, $P^{-1}(y) = \{x \in X : F(T(y), x, y) \not\subseteq C(x)\}$ is open in X.

Indeed, if $x \in \overline{X \backslash P^{-1}(y)}$, then there exists a net $\{x_\alpha\}$ in $P^{-1}(y)$ such that $x_\alpha \to x \in X$. Therefore $F(T(y), x_\alpha, y) \subseteq C(x_\alpha)$ and $x_\alpha \in X$. Let $z \in F(T(y), x, y)$. Since for each fixed $y \in X$, $x \mapsto F(T(y), x, y)$ is lower semicontinuous, there exists a net $\{z_\alpha\}$ in $F(T(y), x_\alpha, y)$ such that $z_\alpha \to z$. Therefore $z_\alpha \in C(x_\alpha)$ for all α. Since $C : X \to 2^{\mathcal{Z}}$ is an upper semicontinuous multivalued map with closed values, it follows that C is closed and thus $z \in C(x)$. Also $x \in X$ and $F(T(y), x, y) \subseteq C(x)$. This shows that $x \in X \backslash P^{-1}(y)$ and hence $X \backslash P^{-1}(y)$ is closed for all $y \in X$. Therefore $P^{-1}(y)$ is open for all $y \in X$.

We define

$$G(x) = \begin{cases} S(x) \cap P(x) & \text{if } x \in \mathcal{F}, \\ S(x) & \text{if } x \notin \mathcal{F}. \end{cases}$$

By (i), $x \notin Q(x) = \text{co}Q(x)$ for all $x \in X$. Therefore $x \notin \text{co}P(x)$ and thus $x \notin \text{co}G(x)$ for all $x \in X$. As in the proofs of Theorems 3.1 and 3.2, we have $G^{-1}(y)$ is open in X for all $y \in X$. Condition (iv) implies for each $x \in X \backslash K$, there exists $\tilde{y} \in M$ such that $x \in G^{-1}(\tilde{y})$. It follows from Theorem 2.2 that there exists $\bar{x} \in X$ such that $G(\bar{x}) = \emptyset$. Since for all $x \in X$, $S(x)$ is nonempty, we have $S(\bar{x}) \cap P(\bar{x}) = \emptyset$ and $\bar{x} \in \mathcal{F}$. From which we have $\bar{x} \in S(\bar{x})$ such that

$$F(s, \bar{x}, y) \subseteq C(\bar{x}) \quad \text{for all } y \in S(\bar{x}) \text{ and } s \in T(y). \tag{8}$$

This completes the proof. □

Corollary 3.3 *Let $L(\mathcal{X}, \mathcal{Z})$ be equipped with the σ-topology. Let $T : X \to L(\mathcal{X}, \mathcal{Z})$ be a multivalued map with nonempty values, $S : X \to 2^X$ a multivalued map with nonempty convex values such that for all $y \in X$, $S^{-1}(y)$ is open in X, and the multivalued map $C : X \to 2^{\mathcal{Z}}$ be upper semicontinuous. Let $g : X \times X \to \mathcal{Z}$ be continuous function such that the following conditions hold.*

(i) $g(x, x) = 0$ *for all* $x \in X$;
(ii) *F is strong type-III maximal C_x-pseudomonotone w.r.t. T, where $F(t, x, y)$ $\langle t, y - x\rangle + g(x, y)$;*
(iii) *For each $(t, x) \in D \times X$, the multivalued map $y \mapsto \langle t, y - x\rangle + g(x, y)$ is C_x-quasiconvex;*
(iv) *There exist a compact subset K of E and a compact convex subset M of X such that for each $x \in X \backslash K$, there exists $y \in M \cap S(x)$ such that $\langle s, y - x\rangle + g(x, y) \notin C(x)$ for some $s \in T(y)$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in S(\bar{x})$ and $\langle t, y - \bar{x}\rangle + g(\bar{x}, y) \in C(\bar{x})$ for all $y \in S(\bar{x})$ and for all $t \in T(y)$.

4 Conclusions

In this paper, we considered generalized vector quasi-equilibrium problems for multivalued maps. We introduced the concept of ℓ-hemicontinuity and several pseudomonotonicities. Under the assumptions of some kind of pseudomonotonicity and u-hemicontinuity/ℓ-hemicontinuity, we established some existence results for solutions of our problems. Many existence results for solutions of the vector quasi-equilibrium problems appeared in the literature can be easily obtained from the results of this paper. By using the technique of Ansari and Flores-Bazán (2003), Ansari and Yao (2003) and Lee (2000) and the results of this paper, the existence results of solutions of vector quasi-optimization problems and vector quasi-saddle point problems for nondifferentiable and nonconvex functions can be derived.

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