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Selecting families and their applications

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Abstract

Deguire and Lassonde [P. Deguire, M. Lassonde, *Familles sélectantes*, Topol. Methods Nonlinear Anal. 5 (1995) 261–269] extend the concept of continuous selection and introduce the notion of selecting family for a family of set-valued mappings. In this paper, we first establish a new existence theorem of selecting families. The existence of the selecting families will be then used in order to obtain several fixed component theorems of Fan–Browder type, an intersection result, a maximal element theorem for a family of set-valued mappings and a minimax inequality.

Keywords: Set-valued mapping; Selecting family; Fan–Browder fixed point theorem; Minimax inequality; Maximal element theorem

1. Introduction

Let X and Y be two nonempty sets and $T : X \multimap Y$ be a set-valued mapping (simply, a map), that is a function that assigns to each $x \in X$, a unique subset $T(x)$ of Y. For each $y \in Y$, the set $T^-(y)$ is called the *fiber* of T at the point y.

It is well-known that any map from a paracompact space to a convex space has a continuous selection whenever it has nonempty convex values and open fibers. This fact was first used by Browder [1,2] in order to establish the so-called Fan–Browder fixed point theorem. Later, it was explicitly formulated by Ben-El-Mechaiekh, Deguire, and Granas [3,4] and by Yannelis and Prabhakar [5], and has been applied by many authors.

In [6] Deguire and Lassonde extend the concept of continuous selection introducing the notion of selecting family for a family of maps. Let us recall this notion.

Definition 1. Let $\mathcal{T} = \{T_i : X \multimap Y_i\}_{i \in I}$ be a family of maps, where X and Y_i $(i \in I)$ are topological spaces. A *selecting family* for $\mathcal{T}$ is a family of continuous functions $\{f_i : X \to Y_i\}_{i \in I}$ satisfying the following condition: for each $x \in X$, there exists $i \in I$ such that $f_i(x) \in T_i(x)$.

One easily observes that the notion of selecting family reduces to the concept of continuous selection, when I has only one element.

The two authors mentioned above, establish the following theorem on the existence of selecting families:

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Theorem 1. *Let X be a paracompact space and $\{Y_i\}_{i\in I}$ a family of convex sets each in a Hausdorff topological vector space. Suppose that $\mathcal{T} = \{T_i : X \multimap Y_i\}_{i\in I}$ is a family of maps satisfying the following conditions:*

(i) *each T_i has convex values;*
(ii) *each T_i has open fibers;*
(iii) *for each $x \in X$, there exists $i \in I$ such that $T_i(x) \neq \emptyset$.*

Then $\mathcal{T}$ has a selecting family.

By Theorem 1, Lassonde and Deguire obtain the following fixed component theorem of Fan–Browder type:

Theorem 2. *Let $\{X_i\}_{i\in I}$ be a family of compact convex sets each in a Hausdorff topological vector space and $\{T_i : X = \prod_{j\in I} X_j \multimap X_i\}_{i\in I}$ a family of maps satisfying conditions* (i), (ii) *and* (iii) *in Theorem* 1. *Then there exist $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ and $i \in I$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$.*

In this paper, we first establish a new existence theorem of selecting families replacing conditions (ii) and (iii) in Theorem 1 by a unique weaker condition. The existence of the selecting families will be then used in order to obtain several fixed component theorems of Fan–Browder type (closely related to Theorem 2), an intersection result, a maximal element theorem for a family of maps and minimax inequalities. The maximal element theorem in this paper is different from Corollary 4.4 in [7], Theorem 4.1 in [8] and many other results concerning maximal elements from the recent literature.

2. Selecting families, fixed component theorems

Theorem 3. *Let X be a paracompact space and $\{Y_i\}_{i\in I}$ a family of convex sets each in a Hausdorff topological vector space. Suppose that $\mathcal{T} = \{T_i : X \multimap Y_i\}_{i\in I}$ is a family of maps with convex values satisfying the condition $X = \bigcup_{i\in I} \bigcup_{y_i\in Y_i} \operatorname{int} T_i^-(y_i)$. Then $\mathcal{T}$ has a selecting family.*

Proof. For each $i \in I$ let $F_i : X \multimap Y_i$ be defined by

$$F_i(x) = \{y_i \in Y_i : x \in \operatorname{int} T_i^-(y_i)\} \quad \text{for all } x \in X.$$

Then $F_i^-(y_i) = \operatorname{int} T_i^-(y_i)$ and $F_i^-(y_i)$ is open. Since $\bigcup_{i\in I} \bigcup_{y_i\in Y_i} \operatorname{int} T_i^-(y_i) = X$, for each $x \in X$, there exist $i \in I$ and $y_i \in Y_i$ such that $x \in \operatorname{int} T_i^-(y_i)$. This shows that $y_i \in F_i(x) \subset \operatorname{co} F_i(x)$, hence $\operatorname{co} F_i(x) \neq \emptyset$. It is easy to see that $F_i(x) \subset T_i(x)$ for all $x \in X$. Let $H_i : X \multimap Y_i$ be defined by $H_i(x) = \operatorname{co} F_i(x)$ for $x \in X$. Then, $H_i(x) \subset \operatorname{co} T_i(x) = T_i(x)$.

Since the maps F_i have open fibers, by Lemma 5.1 in [5], $H_i^-(y) = (\operatorname{co} F_i)^-(y_i)$ is open for each $i \in I$ and $y_i \in Y_i$. Then by Theorem 1, there exists a family of continuous functions $\{f_i : f_i : X \to Y_i\}$ such that for each $x \in X$, there exists $i \in I$ such that $f_i(x) \in H_i(x) \subset T_i(x)$. Hence $\{f_i\}_{i\in I}$ is a selecting family of $\mathcal{T}$. □

It is easy to show that $X = \bigcup_{i\in I} \bigcup_{y_i\in Y_i} \operatorname{int} T_i^-(y_i)$ whenever conditions (ii) and (iii) in Theorem 1 are fulfilled. The following example shows that sometimes we can apply Theorem 3 to show that a family of maps $\mathcal{T}$ has a selecting family, but Theorem 1 is not applicable.

Example. Let $T_1, T_2 : [0, 1] \multimap [0, 1]$ be the maps defined by

$$T_1(x) = \begin{cases} \{1\}, & \text{if } x = 0 \\ (0, x], & \text{if } x \in (0, 1]. \end{cases}$$

$$T_2(x) = \begin{cases} [x, 1), & \text{if } x \in [0, 1) \\ \{0\}, & \text{if } x = 1. \end{cases}$$

Then

$$\bigcup_{y_1\in[0,1]} \operatorname{int} T_1^-(y_1) = (0, 1], \qquad \bigcup_{y_2\in[0,1]} \operatorname{int} T_2^-(y_2) = [0, 1)$$

hence $\bigcup_{i=1}^2 \bigcup_{y_i\in[0,1]} \operatorname{int} T_i^-(y_i) = [0, 1]$. So T_1, T_2 satisfy the requirements of Theorem 3. But $T_1^-(1) = T_2^-(0) = \{0, 1\}$ are not open, hence T_1, T_2 not satisfy all the conditions of Theorem 1.

Observe that, if we denote by $1_{[0,1]}$ the identity function on [0,1], then the family $\{1_{[0,1]}, 1_{[0,1]}\}$ is a selecting family for $\{T_1, T_2\}$.

Recall that if Y is topological space, a map $T : X \multimap Y$ is said to be compact if $T(X)$ is a relatively compact subset of Y.

In the proof our theorem, we need the following particular form of the Himmelberg fixed point theorem [9].

Lemma 4. *If X is a convex subset of a locally convex topological vector space, then any compact continuous function $f : X \to X$ has a fixed point.*

In the next theorems, for a subset X of a topological space and a map $T : X \multimap Y$, by $\operatorname{int} T^-(y)$ we denote the interior of $T^-(y)$ relative to X.

Theorem 5. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff locally convex topological vector space and $\{T_i : X = \prod_{k\in I} X_k \multimap X_i\}_{i\in I}$ a family of compact maps with convex values satisfying the condition $X = \bigcup_{i\in I} \bigcup_{y_i\in X_i} \operatorname{int} T_i^-(y_i)$. Then there exist $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ and $i \in I$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$.*

Proof. For each $i \in I$, $Y_i = \overline{T_i(X)}$ is a compact subset of X_i. Since $Y = \prod_{i\in I} Y_i$ is compact in X, $\widetilde{Y} = \operatorname{co} Y$ is paracompact in X, by Lemma 1 in [10]. By Theorem 3 there exists a selecting family $\{f_i : \widetilde{Y} \to X_i\}_{i\in I}$ for the family of maps $\{T_{i|\widetilde{Y}} : \widetilde{Y} \to X_i\}_{i\in I}$. The function $f : \widetilde{Y} \to \widetilde{Y}$ defined by

$$f(x) = (f_i(x))_{i\in I}, \quad \text{for } x = (x_i)_{i\in I} \in \widetilde{Y},$$

is continuous and compact (since each of its components is compact). By Lemma 4, f has a fixed point $\widetilde{x} = (\widetilde{x}_i)_{i\in I}$. From the definition of the selecting family, there is an index $i \in I$ such that $\widetilde{x}_i = f_i(\widetilde{x}) \in T_i(\widetilde{x})$. □

It would be of some interest to compare Theorem 5 to Theorem 2 in [11].

As we have already remarked, $X = \bigcup_{i\in I} \bigcup_{y_i\in X_i} \operatorname{int} T_i^-(y_i)$ whenever conditions (ii) and (iii) in Theorem 1 are fulfilled. Thus, in locally convex topological vector spaces Theorem 2 remains valid if the compactness condition for the sets X_i is replaced by the compactness of the maps T_i. More precisely we have:

Corollary 6. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff topological locally convex vector space and $\{T_i : X = \prod_{k\in I} X_k \multimap X_i\}_{i\in I}$ a family of compact maps satisfying conditions* (i), (ii) *and* (iii) *of Theorem* 1. *Then there exist $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ and $i \in I$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$.*

When I is a singleton, Theorem 5 reduces to the following corollary:

Corollary 7. *Let X be a convex set in a Hausdorff locally convex topological vector space and $T : X \multimap X$ a compact map with nonempty convex values satisfying the condition $X = \bigcup_{x\in X} \operatorname{int} T^-(x)$. Then there exists $\widetilde{x} \in X$ such that $\widetilde{x} \in T(\widetilde{x})$.*

Taking into account Proposition 1 in [12] we can observe that Corollary 7 coincides with Corollary 3 in [11].

As a consequence of Theorem 5, we obtain a new fixed component theorem.

Theorem 8. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff locally convex topological vector space, $\{Y_j\}_{j\in J}$ a family of nonempty sets, $\{T_i : X = \prod_{k\in I} X_k \multimap X_i\}_{i\in I}$ a family of compact maps and $\{F_j : X \multimap Y_j\}_{j\in J}$ a family of maps such that*

(i) *for each $i \in I$, T_i has convex values;*
(ii) *for each $j \in J$, F_j has open fibers;*
(iii) *for each $x \in X$, there exists $j \in J$ such that $F_j(x) \neq \emptyset$;*
(iv) *for each $j \in J$ and any $y_j \in Y_j$, there exist $i \in I$ and $x_i \in X_i$ such that $F_j^-(y_j) \subset T_i^-(x_i)$.*

Then there exist $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ and $i \in I$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$.

Proof. According to Theorem 5 it suffices to show that
$X = \bigcup_{i\in I} \bigcup_{z_i\in X_i} \operatorname{int} T_i^-(z_i)$.

Let $x \in X$. By (iii) and (iv) there exist $j \in J$, $y_j \in Y_j$, $i \in I$ and $z_i \in X_i$ and such that $x \in F_j^-(y_j) \subset T_i^-(z_i)$. Since $F_j^-(y_j)$ is open, it follows that $x \in \operatorname{int} T_i^-(z_i)$. Thus the proof is complete. □

It is easy to see that Theorem 8 is equivalent to the following maximal element theorem for a family of maps.

Theorem 9. *Let $\{X_i\}_{i\in I}$ be a family of convex sets each in a Hausdorff locally convex topological space, $\{Y_j\}_{j\in J}$ a family of nonempty sets, $\{T_i : X = \prod_{i\in I} X_i \multimap X_i\}_{i\in I}$ a family of compact maps and $\{F_j : X \multimap Y_j\}_{j\in J}$ a family of maps such that*

(i) *for each $i \in I$, T_i has convex values;*
(ii) *for each $j \in J$, F_j has open fibers;*
(iii) *for each $x = (x_i)_{i\in I} \in X$ and $i \in I$, $x_i \notin T_i(x)$;*
(iv) *for each $j \in J$ and any $y_j \in Y_j$, there exist $i \in I$ and $x_i \in X_i$ such that $F_j^-(y_j) \subset T_i^-(x_i)$.*

Then there exists $\bar{x} \in X$ such that $F_j(\bar{x}) = \emptyset$ for all $j \in J$.

Remark 1. Theorem 9 is different from Corollary 4.4 in [7], Theorem 4.1 in [8] and any other existing maximal element theorem from the recent literature. This theorem will have many applications.

The origin of the next theorem goes back to Fan's section lemma (Lemma 4 in [13]).

Theorem 10. *Let $\{X_i\}_{i\in I}$ be a family of compact convex sets each in a Hausdorff locally convex topological vector space, $X = \prod_{i\in I} X_i$ and, for each $i \in I$, $A_i \subset X \times X_i$. Suppose that the following conditions are satisfied:*

(i) *for each $x = (x_i)_{i\in I} \in X$ and any $i \in I$, $(x, x_i) \in A_i$;*
(ii) *for each $x \in X$ and any $i \in I$, the set $\{y_i \in X_i : (x, y_i) \notin A_i\}$ is convex (possibly empty);*
(iii) *for each $i \in I$, there exists a compact subset K_i of X_i such that $X \times (X_i \setminus K_i) \subset A_i$.*

Then, $\bigcap_{i\in I} \bigcap_{y_i\in X_i} \overline{\{x \in X : (x, y_i) \in A_i\}} \neq \emptyset$.

Proof. For each $i \in I$, define the map $T_i : X \multimap X_i$ by

$$T_i(x) = \{y_i \in X_i : (x, y_i) \notin A_i\}.$$

By (iii) it follows at once that $T_i(X) \subset K_i$, hence the maps T_i are compact, have convex values and, by (i), they do not satisfy the conclusion of Theorem 5. Consequently, $\bigcup_{i\in I} \bigcup_{x_i\in X_i} \operatorname{int} T_i^-(x_i) \neq X$. This means that there exists $\widetilde{x} \in X$ such that for each neighborhood V of $\widetilde{x}$ and any index $i \in I$, $\bigcap_{x\in V} T_i(x) = \emptyset$.

Let $i \in I$ and $y_i \in X_i$ be arbitrarily fixed. For each neighborhood V of $\widetilde{x}$, there is a point $x_V \in V$ such that $y_i \notin T_i(x_V)$. If we denote by

$$M(y_i) = \{x \in X : (x, y_i) \in A_i\},$$

it follows that $x_V \in V \cap M(y_i)$, whence $\widetilde{x} \in \overline{M(y_i)}$. Since $i \in I$ and $y_i \in X_i$ were arbitrarily chosen, it follows that

$$\widetilde{x} \in \bigcap_{i\in I} \bigcap_{y_i\in X_i} \overline{\{x \in X : (x, y_i) \in A_i\}}. \quad \square$$

Remark 2. Condition (iii) in Theorem 10 is trivially fulfilled when X_i are all compact.

3. A minimax inequality

Definition 2. Let X be topological space and Y nonempty set. A function $f : X \times Y \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is said to be *transfer upper semicontinuous* on X (see [14]) if for each $\lambda \in \mathbb{R}$ and all $x \in X$, $y \in Y$ with $f(x, y) < \lambda$, there exist a neighborhood $V(x)$ of x and $y' \in Y$ such that $f(x', y') < \lambda$ for all $x' \in V(x)$.

It is clear that every function upper semicontinuous on X is transfer upper semicontinuous on X, but the converse is not true (see [14]).

The following result is a "multiplied" version of Theorem 11 in [15].

Theorem 11. *Let $\{X_i\}_{i\in I}$ be a family of compact convex sets each in a Hausdorff locally convex topological vector space, $X = \prod_{i\in I} X_i$, $\{Y_j\}_{j\in J}$ be a family of nonempty sets and $\{f_i : X \times X_i \to \overline{\mathbb{R}}\}_{i\in I}$, $\{g_j : X \times Y_j \to \overline{\mathbb{R}}\}_{j\in J}$ two families of functions. Suppose that*

(i) *for each* $i \in I$, f_i *is quasiconvex on* X_i*;*
(ii) *for each* $j \in J$, g_j *is transfer upper semicontinuous on* X*;*
(iii) *for each* $j \in J$ *and any* $y_j \in Y_j$*, there exist* $i \in I$ *and* $x_i \in X_i$ *such that* $f_i(., x_i) \leq g_j(., y_j)$.

Then, $\inf_{x\in X} \inf_{i\in I} f_i(x, x_i) \leq \sup_{x\in X} \inf_{j\in J} \inf_{y_j\in Y_j} g_j(x, y_j)$.

Proof. We may suppose that $\sup_{x\in X} \inf_{j\in J} \inf_{y_j\in Y_j} g_j(x, y_j) < \infty$. Let $\lambda > \sup_{x\in X} \inf_{j\in J} \inf_{y_j\in Y_j} g_j(x, y_j)$, be arbitrarily fixed. For all $i \in I$ and $j \in J$, we define the maps $T_i : X \multimap X_i$ $G_j : Y_j \multimap X$ and $F_j : X \multimap Y_j$ by

$$T_i(x) = \{z_i \in X_i : f_i(x, z_i) < \lambda\},$$
$$G_j(y_j) = \{x \in X : g_j(x, y_j) < \lambda\}, \qquad F_j(x) = \{y_j \in Y_j : x \in \operatorname{int} G_j(y_j)\}.$$

We prove that the families of maps $\{T_i\}_{i\in I}$ and $\{F_j\}_{j\in J}$ satisfy all the requirements of Theorem 8. For all $i \in I$, since f_i is quasiconvex on X_i, T_i has convex values. For $j \in J$ and $y_j \in Y_j$, we have $F_j^-(y_j) = \operatorname{int} G_j(y_j)$, hence the maps F_j have open fibers.

Let $x \in X$. We show that $F_j(x) \neq \emptyset$, for some $j \in J$. From $\lambda > \sup_{x\in X} \inf_{j\in J} \inf_{y_j\in Y_j} g_i(x, y_j)$, it follows that there exist $j \in J$ and $y_j \in Y_j$ such that $g_j(x, y_j) < \lambda$. Since g_j is transfer upper semicontinuous on X there exist a neighborhood $V(x)$ of x and $y'_j \in Y_j$ such that $g_j(x', y'_j) < \lambda$ for all $x' \in V(x)$. It follows that $V(x) \subset G_j(y'_j)$. Then $x \in \operatorname{int} G_j(y'_j)$, hence $y'_j \in F_j(x)$.

By (iii), for each $j \in J$ and any $y_j \in Y_j$, there exist $i \in I$ and $x_i \in X_i$ such that $G_j(y_j) \subset T_i^-(x_i)$. Since $F_j^-(y_j) = \operatorname{int} G_j(y_j)$, it follows that $F_j^-(y_j) \subset T_i^-(x_i)$.

Thus all the requirements of Theorem 8 are fulfilled. By Theorem 8, there exist $\widetilde{x} = (\widetilde{x}_i)_{i\in I} \in X$ and $i \in I$ such that $\widetilde{x}_i \in T_i(\widetilde{x})$. Hence

$$\inf_{x\in X} \inf_{i\in I} f_i(x, x_i) \leq f_i(\widetilde{x}, \widetilde{x}_i) < \lambda,$$

which proves the theorem. □

The particular case of the previous theorem $I = J$, $X_i = Y_i$, $f_i = g_i$ for all $i \in I$, is a generalization of Theorem 9 in [15], which is in turn a generalization of the famous Ky Fan minimax inequality [16].

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