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Simultaneous variational relation problems and related applications

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ABSTRACT

The aim of this paper is to present an existence result of the simultaneous variational relation problem. As applications of our result, we study the existence theorems of solution for the equilibrium problem, variational inclusions problem, common fixed points and the minimax theorem.

1. Introduction

Let X, Y and Z be nonempty sets. $S_i : X \to 2^X$, $S_i : X \to 2^Y$ and $H : X \times Y \to 2^Z$. Let R be a relation linking $x \in X$, $y \in Y$ and Z. Luc [1,2] studied the following variational relation problem.

(VR) Find $\bar{x} \in X$ such that $\bar{x} \in S_1(\bar{x})$ and $R(\bar{x}, y, v)$ holds for every $y \in S_2(\bar{x})$ and any $v \in H(\bar{x}, y)$.

(VR) contains optimization problems, variational inclusion problems, differential inclusion problems and equilibrium problems as special cases. In this paper, let X, Y be two nonempty compact convex metrizable subsets in two locally convex topological vector spaces (in short, t.v.s.) and Z be a topological space (in short, t.s.). $S : X \to 2^X$ and $T : X \to 2^Y$ be two multivalued maps with nonempty values. We consider the following problem:

(SVR) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$Q(\bar{x}, \bar{y}, u) \text{ holds for all } u \in S(\bar{x})$$

and

$$R(\bar{x}, \bar{y}, v) \text{ holds for all } v \in T(\bar{x}),$$

where R and Q are relations defined on $X \times Y \times Y$ and $X \times Y \times X$, respectively.

This problem was called a simultaneous variational relation problem in which S, T are constraints and R, Q are variational relations. As applications of our existence result of the simultaneous variational relation problem above, we study the following problems:

(i) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \subseteq G(\bar{x}, \bar{y}, u) \quad \text{for all } u \in S(\bar{x})$$

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and

$$A(\bar{x}, \bar{y}, v) \subseteq B(\bar{x}, \bar{y}, v) \quad \text{for all } v \in T(\bar{x}).$$

(ii) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \bigcap G(\bar{x}, \bar{y}, u) \neq \emptyset \quad \text{for all } u \in S(\bar{x}),$$

and

$$A(\bar{x}, \bar{y}, v) \bigcap B(\bar{x}, \bar{y}, v) \neq \emptyset \quad \text{for all } v \in T(\bar{x}).$$

(iii) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}) \bigcap M(\bar{x}), \bar{y} \in T(\bar{x})$,

and $R(\bar{x}, \bar{y}, v)$ holds for all $v \in T(\bar{x})$.

(iv) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$

$$\text{and } \min_{v \in T(\bar{x})} \max_{u \in \operatorname{cl} S(\bar{x})} h(u, v) = \max_{u \in \operatorname{cl} S(\bar{x})} \min_{v \in T(\bar{x})} h(u, v) = h(\bar{x}, \bar{y}).$$

The problems (i) and (ii) are simultaneous variational inclusion problems. When $A = B$, problems (i) and (ii) were different from the problems studied by Hai et al. [3–5]. In [3–5], Hai et al. studied the variational inclusion problems with a fixed point theorem or maximal element theorems, in this paper, we study the variational inclusion problems with an existence theorem of simultaneous of variational relation problems, the results and techniques are quite different. As special cases of problems (i) and (ii), we consider several simultaneous quasi-equilibrium problems and simultaneous quasivariational inclusion problems:

Let Z be a real t.v.s. and $C : X \to 2^Z$ be a multivalued map such that for each $x \in X$, $C(x)$ is a nonempty closed convex cone with nonempty interior.

(SVEP1) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$A(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x});$$

(SVEP2) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \bigcap C(\bar{x}) \neq \emptyset \quad \text{for all } u \in S(\bar{x})$$

and

$$A(\bar{x}, \bar{y}, v) \bigcap C(\bar{x}) \neq \emptyset \quad \text{for all } v \in T(\bar{x});$$

(SVEP3) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \bigcap \operatorname{int} C(\bar{x}) = \emptyset \quad \text{for all } u \in S(\bar{x})$$

and

$$A(\bar{x}, \bar{y}, v) \bigcap \operatorname{int} C(\bar{x}) = \emptyset \quad \text{for all } v \in T(\bar{x});$$

(SVEP4) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \nsubseteq \operatorname{int} C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$A(\bar{x}, \bar{y}, v) \nsubseteq \operatorname{int} C(\bar{x}) \quad \text{for all } v \in T(\bar{x});$$

(SVEP1, 2, 3, 4) were studied by Lin [6].

If we let $F(x, y, u) = \{0\}$ for all $(x, y, u) \in X \times Y \times X$, then (SVEP1) is reduced to the vector equilibrium problem:

(VEP1) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$ and

$$A(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x});$$

(SVEP2) is reduced to the vector equilibrium problem:

(VEP2) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x}), \bar{y} \in T(\bar{x})$ and

$$A(\bar{x}, \bar{y}, v) \bigcap C(\bar{x}) \neq \emptyset \quad \text{for all } v \in T(\bar{x});$$

(SVEP3) is reduced to the vector equilibrium problem:

(VEP3) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$A(\bar{x},\bar{y},v) \bigcap \operatorname{int} C(\bar{x}) = \emptyset \quad \text{for all } v \in T(\bar{x});$$

(SVEP4) is reduced to the vector equilibrium problem:

(VEP4) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$A(\bar{x},\bar{y},v) \not\subseteq \operatorname{int} C(\bar{x}) \quad \text{for all } v \in T(\bar{x})$$

(VEP1, 2, 3, 4) were studied by Lin et al. [7–9], Sach [10] and references therein.

Similarly, if we let $A(x,y,v) = \{0\}$ for all $(x,y,v) \in X \times Y \times Y$, then (SVEP1, 2, 3, 4) are is reduced to the vector equilibrium problem recently studied by [11,12].

Let $H : X \times Y \times Y \to 2^Z$ and $M : X \times Y \times X \to 2^Z$ be multivalued maps with nonempty values.

The special cases of problem (i) are following simultaneous variational inclusion problems:

(SVIP1) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$M(\bar{x},\bar{y},\bar{x}) \subseteq M(\bar{x},\bar{y},u) - C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$H(\bar{x},\bar{y},\bar{y}) \subseteq H(\bar{x},\bar{y},v) - C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(SVIP2) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$M(\bar{x},\bar{y},u) \subseteq M(\bar{x},\bar{y},\bar{x}) + C(\bar{x}) \quad \text{for all } u \in S(\bar{x})$$

and

$$H(\bar{x},\bar{y},v) \subseteq H(\bar{x},\bar{y},\bar{y}) + C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(SVIP1, 2) were studied in [9,13] and references therein.

The special cases of problem (SVIP1) are the following variational inclusion problems:

(VIP1) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
and $M(\bar{x},\bar{y},\bar{y}) \subseteq M(\bar{x},\bar{y},u) - C(\bar{x})$ for all $u \in S(\bar{x})$.

(VIP2) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
and $H(\bar{x},\bar{y},\bar{y}) \subseteq H(\bar{x},\bar{y},v) - C(\bar{x})$ for all $v \in T(\bar{x})$.

The special cases of problem (SPVIP2) are the following variational inclusion problem:

(VIP3) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
and $M(\bar{x},\bar{y},u) \subseteq M(\bar{x},\bar{y},\bar{x}) + C(\bar{x})$ for all $u \in S(\bar{x})$.

(VIP4) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
and $H(\bar{x},\bar{y},v) \subseteq H(\bar{x},\bar{y},\bar{y}) + C(\bar{x})$ for all $v \in T(\bar{x})$.

(VIP1, 3) were studied in [2,13] and references therein, (VIP2, 4) were studied in [9] and references therein.

The special cases of (SVIP3) are the following variational inclusion problems:

(SVIP4) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$0 \in G(\bar{x},\bar{y},u) \quad \text{for all } u \in S(\bar{x})$$

and

$$0 \in B(\bar{x},\bar{y},v) \quad \text{for all } v \in T(\bar{x}).$$

(VIP4) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$0 \in B(\bar{x},\bar{y},v) \quad \text{for all } v \in T(\bar{x}).$$

(VIP5) Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$0 \in G(\bar{x},\bar{y},u) \quad \text{for all } u \in S(\bar{x}).$$

(VIP5) was studied in [12,14] and (VIP4) was studied in [13].

Let $h : X \times Y \to Z$ be a function and C be a nonempty closed convex cone in Z with $\operatorname{int} C \neq \emptyset$. Let the relations Q and R be defined by

$$Q(x,y,u) \text{ holds iff } h(u,y) - h(x,y) \in -C(x)$$

and

$$R(x,y,v) \text{ holds iff } h(x,v) - h(x,y) \in C$$

then (SVR) is reduced to the following vector saddle point problem:

(VSP1) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$h(u, \bar{y}) - h(\bar{x}, \bar{y}) \in -C \quad \text{for all } u \in S(\bar{x})$$

and

$$h(\bar{x}, v) - h(\bar{x}, \bar{y}) \in C \quad \text{for all } v \in T(\bar{x}).$$

Similarly (SVR) contains the following vector saddle point problem:

(VSP2) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$h(u, \bar{y}) - h(\bar{x}, \bar{y}) \notin \operatorname{int} C \quad \text{for all } u \in S(\bar{x})$$

and

$$h(\bar{x}, v) - h(\bar{x}, \bar{y}) \notin -\operatorname{int} C \quad \text{for all } v \in T(\bar{x}).$$

(VSP3) Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$

$$h(u, \bar{y}) - h(\bar{x}, \bar{y}) \in \operatorname{int} C \quad \text{for all } u \in S(\bar{x})$$

and

$$h(\bar{x}, v) - h(\bar{x}, \bar{y}) \in -\operatorname{int} C \quad \text{for all } v \in T(\bar{x}).$$

From the existence theorems of vector saddle point problem (VSP), we study the existence theorem of solution for minimax problem (iv).

The purpose of this paper is to establish an existence theorem of simultaneous variational relation problems, from this existence theorem, we studied existence theorems of simultaneous variational inclusion problems, existence theorems of solutions for vector saddle point problems, minimax theorem and existence theorem of solution for common fixed point theorem and variational relation problem. As we point out in the introduction, our problems contain many problems and many known results as special cases. Our results are different from Luc [1,2] and Lin et al. [15].

2. Preliminaries

Let X and Y be t.s., we denote 2^X the collection of all subsets of X. Let $T : X \to 2^Y$ be a multivalued map, T is said to be closed (resp. open) if $\operatorname{Gr} T = \{(x, y) \in X \times Y : y \in T(x)\}$ is a closed (resp. open) set in $X \times Y$. Let X be a t.v.s., $A \subseteq X$, we denote $\operatorname{co} A$, $\operatorname{cl} A$ as the convex hull, the closure of A, respectively. As for the definition of lower semicontinuous (in short l.s.c.) and upper semicontinuous (in short u.s.c.) of the map $T : X \to 2^Y$, one can refer to [16]. Throughout this paper, all topologies space are assumed to the Hausdorff.

Definition 2.1. Let X, Y, Z be t.s. We denote $R(x, y, z)$ a relation linking $x \in X, y \in Y, z \in Z$. A relation R is called closed at (x, y, z) if for each net $\{(x_\alpha, y_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ converges to (x, y, z) in $X \times Y \times Z$ and $R(x_\alpha, y_\alpha, z_\alpha)$ holds for all $\alpha \in \Lambda$, then $R(x, y, z)$ holds. A relation R defined on $X \times Y \times Z$ is called closed if relation R is closed at every point of $X \times Y \times Z$.

Definition 2.2 (*[17]*). Let X be a convex subset of a t.v.s. E. A multivalued map $F : X \multimap E$ is said to be a KKM map if

$$\operatorname{co} A \subseteq \bigcup_{x \in A} F(x), \quad \text{for each } A \in \langle X \rangle.$$

Definition 2.3. Let X, Y be real t.v.s., $F : X \to 2^Y$ be a multivalued map and C be a nonempty closed convex cone in Y.

(i) F is said to be C-quasiconvex if for all $x_1, x_2 \in X$, $\lambda \in [0, 1]$,

$$\text{either } F(x_1) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + C$$

or

$$F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) + C;$$

(ii) F is said to be C-quasiconcave if for all $x_1, x_2 \in X$, $\lambda \in [0, 1]$,

$$\text{either } F(x_1) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) - C$$

or

$$F(x_2) \subseteq F(\lambda x_1 + (1 - \lambda)x_2) - C.$$

Theorem 2.1 (*[16]*). *Let X and Y be topological spaces, $T : X \to 2^Y$ be a multivalued map.*

(i) *If T is an u.s.c. multivalued map with closed values, then T is closed.*
(ii) *If Y is a compact space and T is closed, then T is u.s.c.*
(iii) *If X is compact and T is an u.s.c. multivalued map with compact values, then $T(X)$ is compact.*

Theorem 2.2 ([18]). *Let X and Y be topological spaces, $T : X \to 2^Y$ be a multivalued map. Then T is l.s.c. at $x \in X$ if and only if for any $y \in T(x)$ and any net $\{x_\alpha\}_{\alpha\in\Lambda}$ in X converges to x, there exists a net $\{y_\alpha\}_{\alpha\in\Lambda}$, $y_\alpha \in T(x_\alpha)$ for all $\alpha \in \Lambda$ with $y_\alpha \to y$.*

The following theorem proposed by Kim and Tan [19] is the main tool in this paper.

Theorem 2.3 ([19]). *Let I be any index set, $X = \prod_{i\in I} X_i$ and $Y = \prod_{i\in I} Y_i$, where X_i and Y_i be nonempty compact convex metrizable subsets of locally convex t.v.s. E_i and H_i, respectively. $S_i : X \to 2^{X_i}$, $T_i : X \to 2^{Y_i}$ and $P_i : X \times Y \to 2^{X_i}$ be multivalued maps. Suppose that*

(i) *For each $x \in X$, $S_i(x)$ is a nonempty convex subset of X_i;*
(ii) *$\mathrm{cl}\, S_i : X \to 2^{X_i}$ is u.s.c.;*
(iii) *T_i is u.s.c. with nonempty closed convex values;*
(iv) *for all $(x, y) \in X \times Y$ and $x = (x_i)_{i\in I}$, $x_i \notin \mathrm{co}\, P_i(x, y)$;*
(v) *for all $y_i \in X_i$, $S_i^-(y_i) = \{x \in X : y_i \in S(x)\}$ and $P_i^-(y_i)$ are open in $X \times Y$ and X, respectively.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that for each $i \in I$, $\bar{x}_i \in \mathrm{cl}\, S_i(\bar{x})$, $\bar{y}_i \in T_i(\bar{x})$ and $S_i(\bar{x}) \bigcap P_i(\bar{x}, \bar{y}) = \emptyset$.

Theorem 2.4 ([17]). *Let E be a t.v.s., $X \subseteq E$ be an arbitrary set, and $G : X \to 2^E$ be a KKM map. If $G(x)$ is closed for all $x \in X$ and $G(x_0)$ is compact for some $x_0 \in X$. Then $\bigcap\{G(x) : x \in X\} \neq \emptyset$.*

Lemma 2.5 ([16]). *Let X and Y be Hausdorff topological spaces, $G : Y \to 2^X$ be a multivalued map and $W : X \times Y \to 2^{\mathbb{R}}$ be a function and $V(y) = \sup_{x\in G(y)} W(x, y)$.*

(i) *If W is lower semicontinuous on $X \times Y$ and G is lower semicontinuous at y_0, then V is lower semicontinuous at y_0;*
(ii) *If W is upper semicontinuous on $X \times Y$, G is upper semicontinuous at y_0 and $G(y_0)$ is compact, then V is upper semicontinuous at y_0.*

3. Main results

In this section, unless otherwise specify, we assume that X, Y are two nonempty compact convex metrizable subsets in two locally convex t.v.s., E and V respectively. $S : X \to 2^X$ and $T : X \to 2^Y$ be multivalued maps, R and Q are relations defined on $X \times Y \times Y$ and $X \times Y \times X$, respectively.

The following theorem is the main result of this paper.

Theorem 3.1. *Suppose that:*

(i) *S is a multivalued map with nonempty convex values and $S^-(y)$ is open for all $y \in X$ and $\mathrm{cl}\, S : X \to 2^X$ is u.s.c.;*
(ii) *T is a continuous multivalued map with nonempty closed convex values;*
(iii) (a) *the relation R is closed;*
  (b) *for each $x \in X$, any finite subset $\{v_1, v_2, \ldots, v_n\}$ of Y and any $y \in \mathrm{co}\,\{v_1, v_2, \ldots, v_n\}$, there exists $j \in \{1, 2, \ldots, n\}$, such that $R(x, y, v_j)$ holds;*
  (c) *for each $(x, v) \in X \times Y$, the set $\{y \in T(x) : R(x, y, v)$ holds$\}$ is convex; and*
(iv) (a) *for each finite set $\{x_1, x_2 \ldots, x_n\} \subset X$, any $x \in \mathrm{co}\,\{x_1, x_2, \ldots, x_n\}$ and $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that $Q(x, y, x_j)$ holds;*
  (b) *for each $u \in X$, $\{(x, y) \in X \times Y : Q(x, y, u)$ does not hold$\}$ is open.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$Q(\bar{x}, \bar{y}, u)$ holds for all $u \in S(\bar{x})$

and

$R(\bar{x}, \bar{y}, v)$ holds for all $v \in T(\bar{x})$.

Proof. Let $H(x) = \{y \in T(x) : R(x, y, v)$ holds for all $v \in T(x)\}$ and $P(x, y) = \{u \in X : Q(x, y, u)$ does not hold$\}$. We want to show that $H(x)$ is nonempty for all $x \in X$. For each $x \in X$, let $R_x : T(x) \to 2^{T(x)}$ be defined by $R_x(v) = \{y \in T(x) : R(x, y, v)$ holds$\}$. By (iii)(b), for each $x \in X$, R_x is a KKM map.

For each $x \in X$ and $v \in Y$, $R_x(v)$ is a closed set. Indeed, let $y \in \overline{R_x(v)}$, then there exists a net $\{y^\alpha\}$ in $R_x(v)$ such that $y^\alpha \to y$. Then $y^\alpha \in T(x)$ and $R(x, y^\alpha, v)$ holds. By (ii) and Theorem 2.1, $y \in T(x)$. By (iii)(a), $R(x, y, v)$ holds and $R_x(v)$ is closed. Since $R_x(v) \subseteq T(x) \subseteq Y$ and Y_i is compact, then $R_x(v)$ is compact. By Theorem 2.4, $\bigcap_{v\in T(x)} R_x(v) \neq \emptyset$. Let $y \in \bigcap_{v\in T(x)} R_x(v)$, then $y \in H(x)$ and $H(x)$ is nonempty. H is closed. Indeed, if $(x, y) \in \overline{\mathrm{Gr}\, H}$, then there exists a net $\{(x^\alpha, y^\alpha)\}$ in $\mathrm{Gr}\, H$ such that

$(x^\alpha, y^\alpha) \to (x, y)$. Therefore, $y^\alpha \in T(x^\alpha)$ and $R(x^\alpha, y^\alpha, v)$ holds for all $v \in T(x^\alpha)$. For each $v \in T(x)$, by Theorem 2.2, there exists a net $\{v^\alpha\}$ such that $v^\alpha \in T(x^\alpha)$, $v^\alpha \to v$ and $R(x^\alpha, y^\alpha, v^\alpha)$ holds. Since T is closed, $y \in T(x)$ and $R(x, y, v)$ holds for all $v \in T(x)$. Hence $(x, y) \in \mathrm{Gr}\, H$ and H is closed. Since H is closed and Y is compact, by Theorem 2.1 H is u.s.c. with closed values. By (iii)(c), $H(x)$ is convex for each $x \in X$. By (iv), it is easy to see that $P^-(u)$ is open for all $u \in X$ and $x \notin \mathrm{co}\, P(x, y)$. Indeed, suppose $x \in \mathrm{co}\, P(x, y)$, then there exists $\{x_1, x_2, \ldots, x_n\} \subseteq P(x, y)$ and $x \in \mathrm{co}\,\{x_1, x_2, \ldots, x_n\}$, by (iv)(a), $Q(x, y, x_j)$ holds for some $j = 1, 2, \ldots, n$. This leads to a contradiction. Then Theorem 3.1 follows from Theorem 2.3. □

For the special cases of Theorem 3.1, we have the following existence theorems for variational relation problems.

Corollary 3.1. *Suppose that conditions* (i), (ii) *and* (iii) *of Theorem* 3.1, *then there exists* $(\bar{x}, \bar{y}) \in X \times Y$ *such that* $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$

and $R(\bar{x}, \bar{y}, v)$ *holds for all* $v \in T(\bar{x})$.

Proof. Suppose $Q(x, y, u)$ holds for all $x \in X$, $y \in Y$ and $u \in S(x)$. Then condition (iv) of Theorem 3.1 is satisfied. Then Corollary 3.1 follows from Theorem 3.1. □

Corollary 3.2. *Suppose that conditions* (i), (ii) *and* (iv) *of Theorem* 3.1, *then there exists* $(\bar{x}, \bar{y}) \in X \times Y$ *such that* $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

and $Q(\bar{x}, \bar{y}, u)$ *holds for all* $u \in S(\bar{x})$.

Proof. Let R be the relation that $R(x, y, v)$ holds for all $x \in X$, $y \in Y$ and $v \in Y$. Then condition (iii) of Theorem 3.1 is satisfied. Corollary 3.2 follows from Theorem 3.1. □

As applications of Theorem 3.1, we have the following existence theorems of simultaneous variational inclusion problems.

Theorem 3.2. *Let* Z *be a t.s.,* $A, B : X \times Y \times Y \to 2^Z$ *and* $F, G : X \times Y \times X \to 2^Z$ *be multivalued maps. Suppose conditions* (i) *and* (ii) *of Theorem* 3.1 *and suppose that*

(iii) (a) *A is l.s.c., B is closed;*
  (b) *for each $(x, y) \in X \times Y$, any finite subset $\{v_1, v_2, \ldots, v_n\}$ of Y_i, and $y \in \mathrm{co}\,\{v_1, v_2, \ldots, v_n\}$, there exists $j \in \{1, 2, \ldots, n\}$, such that $A(x, y, v_j) \subseteq B(x, y, v_j)$;*
  (c) *for each $(x, v) \in X \times Y$, the set $\{y \in T(x) : A(x, y, v) \subseteq B(x, y, v)\}$ is convex; and*

(iv) (a) *for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, any $x \in \mathrm{co}\,\{x_1, x_2, \ldots, x_n\}$ and $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that $F(x, y, x_j) \subseteq G(x, y, x_j)$;*
  (b) *for each $u \in X$, $(x, y) \to F(x, y, u)$ is l.s.c. and $(x, y) \to G(x, y, u)$ is closed.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \subseteq G(\bar{x}, \bar{y}, u) \quad \text{for all } u \in S(\bar{x})$$

and

$$A(\bar{x}, \bar{y}, v) \subseteq B(\bar{x}, \bar{y}, v) \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let relations Q and R be defined by

$$Q(x, y, u) \text{ holds if and only if } F(x, y, u) \subseteq G(x, y, u)$$

and

$$R(x, y, v) \text{ holds if and only if } A(x, y, v) \subseteq B(x, y, v).$$

By (iii)(a), $R(x, y, v)$ is closed. Indeed, let $\{(x^\alpha, y^\alpha, v^\alpha)\}$ be any net in $X \times Y \times Y$ such that $(x^\alpha, y^\alpha, v^\alpha) \to (x, y, v)$ and $R(x^\alpha, y^\alpha, v^\alpha)$ holds for all $\alpha \in \Lambda$. Then

$$A(x^\alpha, y^\alpha, v^\alpha) \subseteq B(x^\alpha, y^\alpha, v^\alpha) \quad \text{for all } \alpha \in \Lambda.$$

Let $w \in A(x, y, v)$. Since A is l.s.c., there exists a net $\{w^\alpha\}_{\alpha\in\Lambda}$ in Z such that $w^\alpha \in A(x^\alpha, y^\alpha, v^\alpha)$ and $w^\alpha \to w$. Since B is closed, $w \in B(x, y, v)$. This shows that $A(x, y, v) \subseteq B(x, y, v)$ and $R(x, y, v)$ holds. Therefore relation R is closed. By (iii)(b), for each $u \in X$, the set

$$M = \{(x, y) \in X \times Y : Q(x, y, u) \text{ holds}\} \text{ is closed.}$$

Indeed, let $(x, y) \in \overline{M}$, then there exists a net $\{(x^\alpha, y^\alpha)\}_{\alpha\in\Lambda}$ in M such that $(x^\alpha, y^\alpha) \to (x, y)$. One has $(x^\alpha, y^\alpha) \in X \times Y$, $Q(x^\alpha, y^\alpha, u)$ holds for all $\alpha \in \Lambda$. Since $X \times Y$ is a closed set, $(x, y) \in X \times Y$. By (iv) and with the same argument as the proof that R is closed, we can show that $Q(x, y, u)$ holds. Hence $(x, y) \in M$ and M is closed. Therefore, for each $u \in X$, the set

$$\{(x, y) \in X \times Y : Q(x, y, u) \text{ does not hold}\} \text{ is open.}$$

Then Theorem 3.2 follows from Theorem 3.1. □

Remark 3.1. Let Z be a t.v.s. For each $(x, v) \in X \times Y$, $y \to B(x, y, v)$ is concave and $y \to A(x, y, v)$ is convex, then condition (iii)(c) of Theorem 3.2 holds.

Indeed, let $0 \le \lambda \le 1$ and

$$y_1, y_2 \in \{y \in T(x) : A(x, y, v) \subseteq B(x, y, v)\} \quad \text{for some } x \in X,\ v \in Y.$$

Then

$$y_1, y_2 \in T(x), \quad A(x, y_1, v) \subseteq B(x, y_1, v)$$

and

$$A_i(x, y_2, v_i) \subseteq B(x, y_2, v_i).$$

Let $y_\lambda = \lambda y_1 + (1 - \lambda)y_2$, then

$$\begin{aligned} A(x, y_\lambda, v) &\subseteq \lambda A(x, y_1, v) + (1 - \lambda)A(x, y_2, v) \\ &\subseteq \lambda B(x, y_1, v) + (1 - \lambda)B(x, y_2, v) \\ &\subseteq B(x, y_\lambda, v). \end{aligned}$$

We also have $y_\lambda \in T(x)$. This shows that the set

$$\{y \in T(x) : A(x, y, v) \subseteq B(x, y, v)\} \text{ is convex.}$$

Theorem 3.3. *Let Z be a t.s., $A, B : X \times Y \times Y \to 2^Z$ and $F, G : X \times Y \times X \to 2^Z$ be multivalued maps. Suppose conditions* (i) *and* (ii) *of* Theorem 3.1 *and suppose that*

(iii) (a) *A is an u.s.c. multivalued maps with nonempty compact values, B is closed;*
  (b) *for each $(x, y) \in X \times Y$, any finite subset $\{v_1, v_2, \ldots, v_n\}$ of Y, and $y \in \text{co}\{v_1, v_2, \ldots, v_n\}$, there exists $j \in \{1, 2, \ldots, n\}$ such that $A(x, y, v_j) \bigcap B(x, y, v_j) \neq \emptyset$;*
  (c) *for each $(x, v) \in X \times Y$, the set $\{y \in T(x) : A(x, y, v) \bigcap B(x, y, v) \neq \emptyset\}$ is convex; and*

(iv) (a) *for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, any $x \in \text{co}\{x_1, x_2, \ldots, x_n\}$ and $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that $F(x, y, x_j) \bigcap G(x, y, x_j) \neq \emptyset$;*
  (b) *for each $u \in X$, $(x, y) \to F(x, y, u)$ is u.s.c. and $(x, y) \to G(x, y, u)$ is closed.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \text{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$F(\bar{x}, \bar{y}, u) \bigcap G(\bar{x}, \bar{y}, u) \neq \emptyset \quad \textit{for all } u \in S(\bar{x}),$$

and

$$A(\bar{x}, \bar{y}, v) \bigcap B(\bar{x}, \bar{y}, v) \neq \emptyset \quad \textit{for all } v \in T(\bar{x}).$$

Proof. Let relations Q and R be defined by

$$Q(x, y, u) \text{ holds if and only if } F(x, y, u) \bigcap G(x, y, u) \neq \emptyset$$

and

$$R(x, y, v) \text{ holds if and only if } A(x, y, v) \bigcap B(x, y, v) \neq \emptyset.$$

By (iii)(a), R is closed. Indeed, let $\{(x^\alpha, y^\alpha, v^\alpha)\}$ be any net in $X \times Y \times Y$ such that $(x^\alpha, y^\alpha, v^\alpha) \to (x, y, v)$ and $R(x^\alpha, y^\alpha, v^\alpha)$ holds for all $\alpha \in \Lambda$. Then

$$A(x^\alpha, y^\alpha, v^\alpha) \bigcap B(x^\alpha, y^\alpha, v^\alpha) \neq \emptyset \quad \text{for all } \alpha \in \Lambda.$$

Let $w^\alpha \in A(x^\alpha, y^\alpha, v^\alpha) \bigcap B(x^\alpha, y^\alpha, v^\alpha)$.
Let $K = \{x^\alpha : \alpha \in \Lambda\} \bigcup \{x\}$, $L = \{y^\alpha : \alpha \in \Lambda\} \bigcup \{y\}$,

$$\text{and } M = \{v^\alpha : \alpha \in \Lambda\} \bigcup \{v\}.$$

Then K, L and M are compact, hence $K \times L \times M$ is a compact set in $X \times Y \times Y$. By (iii)(a) and Theorem 2.1, $A(K \times L \times M)$ is a compact set and $w^\alpha \in A(K \times L \times M)$, then $\{w^\alpha\}_{\alpha \in \Lambda}$ has a subnet $\{w^{\alpha_\lambda}\}_{\alpha_\lambda \in \Lambda}$ such that $w^{\alpha_\lambda} \to w$ for some $w \in A(x, y, v)$. Since $w^{\alpha_\lambda} \in B(x^{\alpha_\lambda}, y^{\alpha_\lambda}, v^{\alpha_\lambda})$ and B is closed, $w \in B(x, y, v)$. Therefore

$$w \in A(x, y, v) \bigcap B(x, y, v) \neq \emptyset$$

and $R(x, y, v)$ holds. By (iv)(b) and following the same argument as in Theorem 3.2, we can prove that for each $v \in X$, the set

$$\{y \in T(x) : Q(x, y, v) \text{ does not hold}\} \text{ is open.}$$

Then Theorem 3.3 follows from Theorem 3.1. □

For the special case of Theorem 3.3, we have the following corollary.

Corollary 3.3. *Let Z be a t.v.s., $B : X \times Y \times Y \to 2^Z$ and $G : X \times Y \times X \to 2^Z$ be multivalued maps. Suppose conditions* (i) *and* (ii) *of Theorem* 3.1 *and suppose that*

(iii) (a) *B is closed;*
 (b) *for each $(x, y) \in X \times Y$, any finite subset $\{v_1, v_2, \ldots, v_n\}$ of Y, and $y \in \operatorname{co}\{v_1, v_2, \ldots, v_n\}$, there exists $j \in \{1, 2, \ldots, n\}$ such that $0 \in B(x, y, v_j)$;*
 (c) *for each $(x, v) \in X \times Y$, the set $\{y \in T(x) : 0 \in B(x, y, v)\}$ is convex; and*

(iv) (a) *for each finite subset $\{x_1, x_2, \ldots, x_n\} \subset X$, any $x \in \operatorname{co}\{x_1, x_2, \ldots, x_n\}$, and $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that $0 \in G(x, y, x_j)$;*
 (b) *for each $u \in X$, $(x, y) \to G(x, y, u)$ is closed.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$0 \in G(\bar{x}, \bar{y}, u) \quad \text{for all } u \in S(\bar{x}),$$

and

$$0 \in B(\bar{x}, \bar{y}, v) \quad \text{for all } v \in T(\bar{x}).$$

Proof. Let $A : X \times Y \times Y \multimap Z$ and $F : X \times Y \times X \multimap Z$ be defined by

$$A(x, y, v) = \{0\} \quad \text{for all } (x, y, v) \in X \times Y \times Y$$

and

$$F(x, y, u) = \{0\} \quad \text{for all } (x, y, u) \in X \times Y \times X.$$

Then Corollary 3.3 follows from Theorem 3.3. □

Remark 3.2. Let Z be a real t.v.s. and $C : X \to 2^Z$ be a closed multivalued map such that for each $x \in X$, $C(x)$ is a nonempty convex cone.

(i) In Theorem 3.2, if

$$B(x, y, v) = C(x) \quad \text{for all } (x, y, v) \in X \times Y \times Y$$

and

$$G(x, y, u) = C(x) \quad \text{for all } (x, y, u) \in X \times Y \times X.$$

Then Theorem 3.2 is an existence theorem of solution for simultaneous generalized vector quasi-equilibrium problem studied in [6,7,20]:
Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$F(\bar{x}, \bar{y}, u) \subseteq C(\bar{x}) \quad \text{for all } u \in S(\bar{x}),$$
and
$$A(\bar{x}, \bar{y}, v) \subseteq C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(ii) If $\operatorname{int} C(x) \neq \emptyset$ for all $x \in X$ and

$$B(x, y, v) = Z \setminus \operatorname{int} C(x) \quad \text{for all } (x, y, v) \in X \times Y \times Y$$

and

$$G(x, y, u) = Z \setminus \operatorname{int} C(x) \quad \text{for all } (x, y, u) \in X \times Y \times X.$$

Then Theorem 3.2 is an existence theorem of solution for the following problem:
Find $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$F(\bar{x}, \bar{y}, u) \bigcap \operatorname{int} C(\bar{x}) = \emptyset \quad \text{for all } u \in S(\bar{x}),$$
and
$$A(\bar{x}, \bar{y}, v) \bigcap \operatorname{int} C(\bar{x}) = \emptyset \quad \text{for all } v \in T(\bar{x}).$$

(iii) If $B(x,y,v) = H(x,y,v) - C(x)$ for all $(x,y,v) \in X \times Y \times Y$,

$$A(x,y,v) = H(x,y,y) \quad \text{for all } (x,y,v) \in X \times Y \times Y,$$
$$\text{and} \quad G(x,y,u) = M(x,y,u) - C(x) \quad \text{for all } (x,y,u) \in X \times Y \times X,$$
$$F(x,y,u) = M(x,y,x) \quad \text{for all } (x,y,u) \in X \times Y \times X,$$

where $H : X \times Y \times Y \to 2^Z$ and $M : X \times Y \times X \to 2^Z$ are u.s.c. multivalued map with nonempty compact values. Then Theorem 3.2 is an existence theorem of solution for systems of generalized vector quasivariational inclusion problem studied in [2,9,20,21] and reference therein:
Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$M(\bar{x},\bar{y},\bar{x}) \subseteq M(\bar{x},\bar{y},u) - C(\bar{x}) \quad \text{for all } u \in S(\bar{x}),$$
and
$$H(\bar{x},\bar{y},\bar{y}) \subseteq H(\bar{x},\bar{y},v) - C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(iv) If $B(x,y,v) = H(x,y,y) + C(x)$ for all $(x,y,v) \in X \times Y \times Y$,

$$A(x,y,v) = H(x,y,v) \quad \text{for all } (x,y,v) \in X \times Y \times Y,$$
$$\text{and} \quad G(x,y,u) = M(x,y,x) + C(x) \quad \text{for all } (x,y,u) \in X \times Y \times X,$$
$$F(x,y,u) = M(x,y,u) \quad \text{for all } (x,y,u) \in X \times Y \times X,$$

where H, M and C are the same as (iii) above. Then Theorem 3.2 is an existence theorem of solution for the following systems of generalized vector quasivariational inclusion problem studied in [2,20]:
Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$M(\bar{x},\bar{y},u) \subseteq M(\bar{x},\bar{y},\bar{x}) + C(\bar{x}) \quad \text{for all } u \in S(\bar{x}),$$
and
$$H(\bar{x},\bar{y},v) \subseteq H(\bar{x},\bar{y},\bar{y}) + C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

(v) In Theorem 3.3, if

$$B(x,y,v) = C(x) \quad \text{for all } (x,y,v) \in X \times Y \times Y$$

and

$$G(x,y,u) = C(x) \quad \text{for all } (x,y,u) \in X \times Y \times X.$$

Then Theorem 3.3 is an existence theorem of solution for the following systems of generalized vector quasi-equilibrium problem studied in [6,7,20]:
Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$F(\bar{x},\bar{y},u) \bigcap C(\bar{x}) \neq \emptyset \quad \text{for all } u \in S(\bar{x}),$$
and
$$A(\bar{x},\bar{y},v) \bigcap C(\bar{x}) \neq \emptyset \quad \text{for all } v \in T(\bar{x}).$$

(vi) In Theorem 3.3, if

$$B(x,y,v) = Z \setminus -\mathrm{int}\, C(x) \quad \text{for all } (x,y,v) \in X \times Y \times Y$$

and

$$G(x,y,u) = Z \setminus -\mathrm{int}\, C(x) \quad \text{for all } (x,y,u) \in X \times Y \times X.$$

Then Theorem 3.2 is an existence theorem of solution for simultaneous generalized vector quasi-equilibrium problem studied in [6,7]:
Find $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,
$$F(\bar{x},\bar{y},u) \not\subseteq -\mathrm{int}\, C(\bar{x}) \quad \text{for all } u \in S(\bar{x}),$$
and
$$A(\bar{x},\bar{y},v) \not\subseteq -\mathrm{int}\, C(\bar{x}) \quad \text{for all } v \in T(\bar{x}).$$

Theorem 3.4. *Let X be a nonempty compact convex subset of a normed vector space, let $M : X \to 2^X$ be a multivalued map with nonempty compact convex values. Suppose conditions* (i), (ii) *and* (iii) *of* Theorem 3.1 *and suppose that:*

(iv) *M is a continuous multivalued map and $M(x) \bigcap S(x) \neq \emptyset$ for all $x \in X$.*

Then there exists $(\bar{x},\bar{y}) \in X \times Y$ such that $\bar{x} \in \mathrm{cl}\, S(\bar{x}) \bigcap M(\bar{x})$, $\bar{y} \in T(\bar{x})$, and $R(\bar{x},\bar{y},v)$ holds for all $v \in T(\bar{x})$.

Proof. Let the relation Q be defined by

$$Q(x,y,u_i) \text{ holds iff } d(x, M(x)) \le d(u, M(x)).$$

It suffices to show that condition (iv) of Theorem 3.1 is satisfied. Suppose there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, $x \in \mathrm{co}\,\{x_1, x_2, \ldots, x_n\}$ and $y \in Y$ such that $Q(x,y,x_j)$ does not hold for all $j \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned} d(x, M(x)) &> d(x_j, M(x)) \\ &= \inf_{z \in M(x)} \|x_j - z\| = \|x_j - z_j\| \quad \text{for some } z_j \in M(x). \end{aligned}$$

Since $x \in \text{co}\,\{x_1, x_2, \ldots, x_n\}$, there exists $\lambda_1 \geq 0, \ldots, \lambda_n \geq 0$ with $\sum_{j=1}^{n} \lambda_j = 1$ such that $x = \sum_{j=1}^{n} \lambda_j x_j$. By assumption, $M(x)$ is convex for each $x \in X$, $x = \sum_{j=1}^{n} \lambda_j x_j \in M(x)$.

$$\begin{aligned} d(x, M(x)) &> \sum_{j=1}^{n} \lambda_j \|x_j - z_j\| \\ &\geq \left\| \sum_{j=1}^{n} \lambda_j x_j - \sum_{j=1}^{n} \lambda_j z_j \right\| = \left\| x - \sum_{j=1}^{n} \lambda_j z_j \right\| \\ &\geq d(x, M(x)). \end{aligned}$$

This leads to a contradiction. Therefore, for each finite subset $\{x_1, x_2, \ldots, x_n\}$ of X, any $x \in \text{co}\,\{x_1, x_2, \ldots, x_n\}$ and $y \in Y$, there exists $j \in \{1, 2, \ldots, n\}$ such that

$$d(x, M(x)) \leq d(x_j, M(x)).$$

That is $Q(x, y, x_j)$ holds.

Since $M : X \to 2^X$ is a continuous map, it follows from Lemma 2.5 that for each $u \in X$

$$x \to d(x, M(x)) = \inf_{w \in M(x)} d(x, w)$$

and $x \to d(u, M(x))$ are continuous functions.

By the definition of Q, for each $u \in X$,

$$\{(x, y) \in X \times Y : d(x, M(x)) \leq d(u, M(x))\} = \{(x, y) \in X \times Y : Q(x, y, u) \text{ holds}\} \text{ is a closed set in } X \times Y.$$

Hence, the set

$$\{(x, y) \in X \times Y : Q(x, y, u) \text{ does not hold}\} \text{ is open in } X \times Y.$$

Then by Theorem 3.1 that there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \text{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$, $d(\bar{x}, M(\bar{x})) \leq d(u, M(\bar{x}))$ holds for all $u \in S(\bar{x})$ and $R(\bar{x}, \bar{y}, v)$ holds for all $v \in T(\bar{x})$. By (iv), $M(\bar{x}) \bigcap S(\bar{x}) \neq \emptyset$.

Take $\bar{u} \in M(\bar{x}) \bigcap S(\bar{x})$,

then $d(\bar{x}, M(\bar{x})) \leq d(\bar{u}, M(\bar{x})) = 0$ and hence $d(\bar{x}, M(\bar{x})) = 0$.

Since $M(\bar{x})$ is a closed set, $\bar{x} \in M(\bar{x}) \bigcap \text{cl}\, S(\bar{x})$. □

Remark 3.3. Theorem 3.4 is an existence theorem of solution for common fixed point and variational relation problem.

As an application of Theorem 3.1, we study the following existence theorem of solution for vector saddle point.

Theorem 3.5. *Let Z be a real t.v.s. and C be a nonempty closed convex cone in Z. Suppose conditions* (i) *and* (ii) *of* Theorem 3.1 *and suppose that:*

(iii) *$h : X \times Y \to Z$ is a continuous function and for each $x \in X$, $v \to h(x, v)$ is C-quasiconvex;*
(iv) *for each $y \in Y$, $x \to h(x, y)$ is C-quasiconcave.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \text{cl}\, S(\bar{x})$, $\bar{y} \in T(\bar{x})$,

$$h(u, \bar{y}) - h(\bar{x}, \bar{y}) \in -C \quad \textit{for all } u \in S(\bar{x})$$

and

$$h(\bar{x}, v) - h(\bar{x}, \bar{y}) \in C \quad \textit{for all } v \in T(\bar{x}).$$

Proof. Let the relations Q and R be defined by

$$Q(x, y, u) \text{ holds iff } h(u, y) - h(x, y) \in -C$$
$$R(x, y, v) \text{ holds iff } h(x, v) - h(x, y) \in C.$$

Since h is continuous, it is easy to see that conditions (iii)(a) and (iv)(b) of Theorem 3.1 hold.

By the definitions of C-quasiconvex and C-quasiconcave, we see that conditions (iii)(b), (iii)(c) and (iv)(a) of Theorem 3.1 hold. Then Theorem 3.5 follows from Theorem 3.1. □

Theorem 3.5 can also be applied to study the following minimax theorem.

Theorem 3.6. *Suppose conditions* (i) *and* (ii) *of* Theorem 3.1 *and suppose that:*

(iii) $h : X \times Y \to \mathbb{R}$ *is a continuous function and for each* $x \in X$, $v \to h(x, v)$ *is quasiconvex;*
(iv) *for each* $y \in Y$, $x \to h(x, y)$ *is quasiconcave.*

Then there exists $(\bar{x}, \bar{y}) \in X \times Y$ *such that* $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$ *and*

$$\min_{v \in T(\bar{x})} \max_{u \in \operatorname{cl} S(\bar{x})} h(u, v) = \max_{u \in \operatorname{cl} S(\bar{x})} \min_{v \in T(\bar{x})} h(u, v) = h(\bar{x}, \bar{y}).$$

Proof. By Theorem 3.1, there exists $(\bar{x}, \bar{y}) \in X \times Y$ such that $\bar{x} \in \operatorname{cl} S(\bar{x})$, $\bar{y} \in T(\bar{x})$ and

$$h(u, \bar{y}) \le h(\bar{x}, \bar{y}) \le h(\bar{x}, v) \quad \text{for all } u \in S(\bar{x}) \text{ and } v \in T(x).$$

Hence

$$\max_{x \in \operatorname{cl} S(\bar{x})} \min_{y \in T(\bar{x})} h(x, y) \ge \min_{y \in T(\bar{x})} \max_{x \in \operatorname{cl} S(\bar{x})} h(x, y).$$

Since

$$\max_{x \in \operatorname{cl} S(\bar{x})} \min_{y \in T(\bar{x})} h(x, y) \le \min_{y \in T(\bar{x})} \max_{x \in \operatorname{cl} S(\bar{x})} h(x, y)$$

is always true, we have

$$\max_{x \in \operatorname{cl} S(\bar{x})} \min_{y \in T(\bar{x})} h(x, y) = \min_{y \in T(\bar{x})} \max_{x \in \operatorname{cl} S(\bar{x})} h(x, y). \quad \square$$

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