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From an abstract maximal element principle to optimization problems, stationary point theorems and common fixed point theorems

Lai-Jiu Lin · Wei-Shih Du

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Abstract In this paper, we first establish an existence theorem related with intersection theorem, maximal element theorem and common fixed point theorem for multivalued maps by applying an abstract maximal element principle proved by Lin and Du. Some new stationary point theorems, minimization problems, new fixed point theorems and a system of nonconvex equilibrium theorem are also given.

Keywords Sizing-up · μ-Bounded · Intersection theorem · Maximal element · Stationary point theorem · Fixed point theorem · Minimization problem · Dancš–Hegedüs–Medvegyev's principle · Equilibrium theorem

1 Introduction

In the proof of the fundamental theorem of Bishop and Phelps [3] on the density of the set of support points of a closed convex subset of a Banach space, the existence of maximal elements in certain partially ordered complete subsets of a normed linear space played a central role. Subsequently, the existence of maximal elements was extended for various other purposes; see [1, 5–7, 9, 10, 12, 13, 15–17] and references therein. It is well-known that the famous Brézis–Browder's maximal element principle [4] is a powerful tool in the fields of applied mathematical analysis and nonlinear analysis. Various generalizations in different directions of maximal element principle (MEP, for short) have been investigated by several authors in the past. In 1990, Kang and Park [10] proved some maximal element theorems on countably inductive quasi-ordered sets. Granas and Horvath [9] studied on a so-called Cantor space and also obtained maximal element theorems and fixed point theorems which

L.-J. Lin (✉)
Department of Mathematics, National Changhua University of Education, Changhua 50058, Taiwan
e-mail: maljlin@cc.ncue.edu.tw

W.-S. Du
Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan
e-mail: wsdu@nknucc.nknu.edu.tw

can be applied to Ekeland's variational principle and their equivalent formulations in complete metric spaces. Park [15] also gave generalized forms of Ekeland's principle and its six equivalents. In [12], the authors also obtained a vectorial version of Ekeland's variational principle and maximal element theorem, a nonconvex minimax theorem and nonconvex (vectorial) equilibrium theorems. However, few authors are concerned about a sufficient condition for the existence of an upper bound for a nondecreasing sequence on a quasi-ordered set. Motivated by this reason, Lin and Du [13] first introduced notions of sizing-up function and μ-bounded quasi-ordered set to describe and establish abstract MEP, and then they established several different versions of generalized Ekeland's variational principle and MEP; for more detail, see [7, 13].

It is well-known that the primitive Ekeland's variational principle is equivalent to the Caristi's fixed point theorem, to the Takahashi's nonconvex minimization theorem, to the drop theorem, and to the petal theotrm; for detail, one can refer to [7, 9–12, 15, 18]. In 1983, Dancš et al. [6] proved the following existence theorem of stationary points for a generated dynamical system which is forceful tools in applied mathematical analysis.

Dancš–Hegedüs–Medvegyev's principle [6] Let (X, d) be a complete metric space and $\Gamma : X \to 2^X$ a multivalued map with nonempty values. Suppose that the following conditions are satisfied:

(1) for each $x \in X$, we have $x \in \Gamma(x)$ and $\Gamma(x)$ is closed;
(2) $x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$;
(3) for any $\{x_n\} \subset X$ with $x_{n+1} \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Then there exists $v \in X$ such that $\Gamma(v) = \{v\}$.

Recently, Lin and Chuang [14] studied the existence theorem of variational disclusion and inclusion problems in the Ekeland's sense by using Dancš–Hegedüs–Medvegyev's principle. By these existence results, Lin and Chuang studied the existence results of some variants of variational intersection and inclusion problems in the Ekeland's sense and the existence results of variants of set valued vectorial Ekeland's variational principle. Their results and approaches were different from any existence theorems for generalized variational inclusion and disclusion problems. For details, see [14] and references therein.

In this paper, we establish an existence theorem related with intersection theorem, maximal element theorem and common fixed point theorem for multivalued maps by applying an abstract maximal element principle proved by Lin and Du [7, 13]. Some new stationary point Theorems, minimization problems, new fixed point theorems and a system of nonconvex equilibrium theorem in metric spaces and uniform spaces are also given.

2 Preliminaries

The following notations related to binary relations on a nonempty set X will be used in this paper. For subsets V, U of $X \times X$, define

$$\Delta = \{(x, x) : x \in X\} \quad (\text{the } \textit{diagonal} \text{ of } X \times X),$$
$$U[x] = \{y \in X : (x, y) \in U\} \quad (\text{the } \textit{entourage} \text{ of } x \in X),$$
$$U^{-1} = \{(x, y) \in X \times X : (y, x) \in U\}$$

and

$$U \circ V = \{(x, y) \in X \times X : (z, y) \in U \quad \text{and} \quad (x, z) \in V \quad \text{for some} \quad z \in X\}.$$

A *uniform space* $(X, \mathcal{U})$ is a nonempty set X endowed with a uniformity $\mathcal{U}$ and satisfies the following conditions:

$(u1)$ $\Delta \subseteq V$ for any $V \in \mathcal{U}$;
$(u2)$ if $V_1, V_2 \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \subset V_1 \cap V_2$;
$(u3)$ if $V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that $W \circ W^{-1} \subset V$;
$(u4)$ if $V \in \mathcal{U}$ and $V \subset W \subset X \times X$, then $W \in \mathcal{U}$.

Two points x and y of X are said to be V-*close* whenever $(x, y) \in V$ and $(y, x) \in V$. Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integers, respectively. A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is called a *Cauchy sequence* for $\mathcal{U}$ (($\mathcal{U}$)- *Cauchy sequence*, for short) if for any $V \in \mathcal{U}$, there exists $\ell \in \mathbb{N}$ such that x_n and x_m are V-close for $n, m \geq \ell$. A nonempty subset C of X is said to be *sequentially* ($\mathcal{U}$)-*complete* if every ($\mathcal{U}$)-Cauchy sequence in C converges. A uniformity $\mathcal{U}$ defines a unique topology $\tau(\mathcal{U})$ on X. A uniform space $(X, \mathcal{U})$ is said to be *Hausdorff* if and only if the intersection of all the $V \in \mathcal{U}$ reduces to the diagonal Δ of X, that is, if $(x, y) \in V$ for all $V \in \mathcal{U}$ implies $x = y$. This guarantees the uniqueness of limits of sequences.

Let X be a nonempty set and "$\lesssim$ " a quasi-order (preorder or pseudo-order; that is, a reflexive and transitive relation) on X. Then $(X, \lesssim)$ is called a quasi-ordered set. A function $\varphi : X \to (-\infty, \infty]$ is called to be $\lesssim$-*nonincreasing* (resp. *strictly* $\lesssim$-*nonincreasing*) if $x, y \in X$ with $x \lesssim y$ implies $\varphi(x) \geq \varphi(y)$ (resp. $\varphi(x) > \varphi(y)$). An element v in X is called a *maximal element* of X if there is no element x of X, different from v, such that $v \lesssim x$. Let $(X, d, \lesssim)$ be a metric space on which a quasi-order order $\lesssim$ is defined. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called *nondecreasing* (resp. *nonincreasing*) if $x_n \lesssim x_{n+1}$ (resp. $x_{n+1} \lesssim x_n$) for each $n \in \mathbb{N}$. A nonempty subset M of X is said to be *sequentially* $\lesssim$-*complete* if every nondecreasing Cauchy sequence in M converges. Let D be a nonempty subset of a quasi-ordered set $(X, \lesssim)$ with a uniformity $\mathcal{U}$ on X. D is said to be *sequentially* $(\mathcal{U}, \lesssim)$-*complete* if every nondecreasing ($\mathcal{U}$)-Cauchy sequence in D converges.

A point v in X is a fixed point of a multivalued map $T : X \to 2^X$ if $v \in T(v)$. The set of fixed points of T is denoted by $\mathcal{F}(T)$.

The notion of sizing-up function, first introduced in Lin and Du [7, 13], is given below, and examples can be found in [7, 13].

Definition 2.1 [7, 13] Let X be a nonempty set. A function $\mu : 2^X \to [0, \infty]$ defined on the power set 2^X of X is called *sizing-up* if it satisfies the following properties:

$(\mu1)$ $\mu(\emptyset) = 0$;
$(\mu2)$ $\mu(A) \leq \mu(B)$ if $A \subseteq B$.

Now, we introduce the concept of smart sizing-up function.

Definition 2.2 Let X be a nonempty set.

(a) A sizing-up function $\mu : 2^X \to [0, \infty]$ is called *smart* if $(\mu3)$ holds, where

$$(\mu3) \quad \mu(\{x, y\}) > 0 \quad \text{for any} \quad x, y \in X \quad \text{with} \quad x \neq y.$$

Example

(a) Let X be a nonempty set. Define $\mu : 2^X \to [0, \infty]$ by

$$\mu(A) = \begin{cases} 0, & \text{if} \quad A = \emptyset; \\ 1, & \text{if} \quad A \neq \emptyset. \end{cases}$$

Then μ is a smart sizing-up function.

(b) Let X be a finite set with $\sharp(X) \in \mathbb{N}$, where $\sharp(X)$ is the cardinal number of X. Define $\mu : 2^X \to [0, \infty]$ by

$$\mu(A) = \sharp(A) \quad \text{for each } A \in 2^X.$$

Then μ is a smart sizing-up function.

(c) Let (X, d) be a metric space. Then the function $\mu_d : 2^X \to [0, \infty]$ defined by $\mu_d(A) := diam(A)$ (the diameter of $A \subset X$) is a smart sizing-up function.

(d) Let X be a nonempty set and $f : X \to \mathbb{R}$ be a function with $f(x) \neq f(y)$ if $x \neq y$. Then $\mu : 2^X \to [0, \infty]$ defined by

$$\mu(A) = \begin{cases} 0, & \text{if } \ A = \emptyset; \\ \sup\{|f(x) - f(y)| : x, y \in A\}, & \text{if } \ A \neq \emptyset. \end{cases}$$

is a smart sizing-up function.

Definition 2.3 [7, 13] Let X be a nonempty set and $\mu : 2^X \to [0, \infty]$ a sizing-up function. A multivalued map $T : X \to 2^X$ with nonempty values is said to be of *type* (μ) if for each $x \in X$ and $\varepsilon > 0$, there exists a $y = y(x, \varepsilon) \in T(x)$ such that $\mu(T(y)) \leq \varepsilon$.

Definition 2.4 [7, 13] A quasi-ordered set $(X, \lesssim)$ with a sizing-up function $\mu : 2^X \to [0, \infty]$, denoted by $(X, \lesssim, \mu)$, is said to be *μ-bounded* if every nondecreasing sequence $x_1 \lesssim x_2 \lesssim \cdots \lesssim x_n \lesssim x_{n+1} \lesssim \cdots$ in X satisfying

$$\lim_{n\to\infty} \mu(\{x_n, x_{n+1}, \ldots\}) = 0$$

has an upper bound.

3 Nonlinear existence theorems induced by an abstract MEP

The following abstract maximal element principle was established by Lin and Du in [7, 13].

Theorem 3.1 [7, 13] *Let $(X, \lesssim, \mu)$ be a μ-bounded quasi-ordered set with a sizing-up function $\mu : 2^X \to [0, \infty]$. For each $x \in X$, let $S : X \to 2^X$ be defined by $S(x) = \{y \in X : x \lesssim y\}$. If S is of type (μ), then for each $x_0 \in X$, there exists a nondecreasing sequence $x_0 \lesssim x_1 \lesssim x_2 \lesssim \cdots$ in X and $v \in X$ such that*

(i) *v is an upper bound of $\{x_n\}_{n=0}^{\infty}$;*
(ii) *$S(v) \subseteq \bigcap_{n=0}^{\infty} S(x_n)$;*
(iii) *$\mu(\bigcap_{n=0}^{\infty} S(x_n)) = \mu(S(v)) = 0$.*

Applying Theorem 3.1, we first establish the following existence theorem related with intersection theorem, maximal element theorem and common fixed point theorem for multivalued maps.

Theorem 3.2 *Let $(X, \lesssim, \mu)$ be a μ-bounded quasi-ordered set with a sizing-up function $\mu : 2^X \to [0, \infty]$. For each $x \in X$, let $S : X \to 2^X$ be defined by $S(x) = \{y \in X : x \lesssim y\}$. Let Λ be any index set. For each $j \in \Lambda$, let $H_j : X \multimap X$ be a multivalued map with nonempty values such that $H_j(x) \cap S(x) \neq \emptyset$ for all $x \in X$. If S is of type (μ) and assume further that μ is smart, then for each $x_0 \in X$, there exists a nondecreasing sequence $x_0 \lesssim x_1 \lesssim x_2 \lesssim \cdots$ in X and $v \in X$ such that*

(i) *v is an upper bound of* $\{x_n\}_{n=0}^{\infty}$*;*
(ii) $\bigcap_{n=0}^{\infty} S(x_n) = S(v) = \{v\}$*;*
(iii) $\mu(\bigcap_{n=0}^{\infty} S(x_n)) = \mu(S(v)) = 0$.
(iv) *v is a maximal element of X.*
(v) $v \in \bigcap_{j\in\Lambda} H_j(v)$.

Proof By Theorem 3.1, for each $x_0 \in X$, there exists a nondecreasing sequence $x_0 \lesssim x_1 \lesssim x_2 \lesssim \cdots$ in X such that the conclusions (i) and (iii) hold. Moreover, we have,

$$v \in S(v) \subseteq \bigcap_{n=0}^{\infty} S(x_n). \tag{$*$}$$

Now, we claim that $\bigcap_{n=0}^{\infty} S(x_n) = \{v\}$. Suppose that there exists $u \in \bigcap_{n=0}^{\infty} S_n(x_n)$ with $u \neq v$. Thus, by $(\mu 2)$ and $(\mu 3)$, we have,

$$0 < \mu(\{u, v\}) \leq \mu\left(\bigcap_{n=0}^{\infty} S(x_n)\right) = 0,$$

which is a contradiction. Hence $\bigcap_{n=1}^{\infty} S(x_n) = \{v\}$. The conclusion (ii) follows from $(*)$ and $\bigcap_{n=0}^{\infty} S(x_n) = \{v\}$. To see (iv), if $v \lesssim w$ for some $w \in X$ with $w \neq v$, then $x_n \lesssim w$ for all $n \in \mathbb{N} \cup \{0\}$ or $w \in \bigcap_{n=0}^{\infty} S(x_n)$. By $(\mu 2)$ and $(\mu 3)$, it follows that

$$0 < \mu(\{v, w\}) \leq \mu\left(\bigcap_{n=0}^{\infty} S(x_n)\right) = 0,$$

which leads a contradiction. Hence v is a maximal element of X. Since $S(v) \cap H_j(v) \neq \emptyset$ for any $j \in \Lambda$ and $S(v) = \{v\}$, we obtain $v \in \bigcap_{j\in\Lambda} H_j(v)$ and (v) is proved. □

Theorem 3.3 *Let (X, d) be a complete metric space and $\Gamma : X \to 2^X$ a multivalued map with nonempty values. Suppose that the following conditions are satisfied:*

(i) *for each $x \in X$, we have $x \in \Gamma(x)$ and $\Gamma(x)$ is closed;*
(ii) *$x, y \in X$ with $y \in \Gamma(x)$ implies $\Gamma(y) \subseteq \Gamma(x)$;*
(iii) *for any $\{x_n\} \subset X$ with $x_{n+1} \in \Gamma(x_n)$ for each $n \in \mathbb{N}$, we have* $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Then there exists a quasi-order $\lesssim$ and a smart sizing-up function $\mu : 2^X \to [0, \infty]$ such that $(X, \lesssim, \mu)$ is a μ-bounded quasi-ordered set.

Proof Let $\mu_d : 2^X \to [0, \infty]$ be defined by $\mu_d(A) := diam(A)$ for $A \subset X$. Then μ_d is a smart sizing-up function. Define a binary relation $\lesssim_{(\Gamma)}$ on X by

$$x \lesssim_{(\Gamma)} y \iff y \in \Gamma(x).$$

It is easy to see that $\lesssim_{(\Gamma)}$ is a quasi-order from conditions (i) and (ii) and $S(x) = \{y \in X : x \lesssim_{(\Gamma)} y\} = \Gamma(x)$. We claim that S is of type (μ_d). Let $x \in M$ and $\varepsilon > 0$ be given. Then there exists $n_1 \in \mathbb{N}$, such that $\frac{1}{2^{n_1}} < \frac{\varepsilon}{4}$. Define a function $\tau : X \to \mathbb{R}$ by

$$\tau(x) = \inf_{y\in S(x)} [-d(x, y)].$$

Hence

$$0 \leq d(x, y) \leq -\tau(x) \quad \text{for all} \quad y \in S(x). \tag{3.1}$$

Note that $\tau(u) > -\infty$ for some $u \in X$. Indeed, suppose to the contrary that $\tau(x) = -\infty$ for each $x \in X$. Take $z_1 \in X$. Thus $\tau(z_1) = \inf_{y \in S(z_1)}[-d(z_1, y)] < -1$. Hence there exists $z_2 \in S(z_1)$ such that $-d(z_1, z_2) < -1$ or $d(z_1, z_2) > 1$. Since $\tau(z_2) < -2$, there exists $z_3 \in S(z_2)$ such that $d(z_2, z_3) > 2$. Continuing in the process, we obtain a sequence $\{z_n\} \subset X$, such that for each $n \in \mathbb{N}$,

- $z_{n+1} \in S(z_n) = \Gamma(z_n)$;
- $d(z_n, z_{n+1}) > n$.

So, we have $\lim_{n\to\infty} d(z_n, z_{n+1}) = \infty$ which contracts with the condition (iii). Therefore there exists $u \in X$ such that $\tau(u) > -\infty$. Let $x_1 = u$. Choose $x_2 \in S(x_1)$ such that

$$-d(x_1, x_2) \leq \tau(x_1) + \frac{1}{2}.$$

Since $S(x_2) \subseteq S(x_1)$ from (ii), we have

$$\begin{aligned}\tau(x_1) &= \inf_{y \in S(x_1)} [-d(x_1, y)] \\ &\leq \inf_{y \in S(x_2)} [-d(x_1, y)] \\ &\leq \inf_{y \in S(x_2)} [-d(x_2, y) + d(x_2, x_1)] \\ &= \tau(x_2) + d(x_1, x_2).\end{aligned}$$

So $\tau(x_2) > -\infty$. Let $k \in \mathbb{N}$ and assume that $x_k \in X$ is already known. Then, one can choose $x_{k+1} \in S(x_k)$ such that

$$-d(x_k, x_{k+1}) \leq \tau(x_k) + \frac{1}{2^k}.$$

By induction, we obtain a nondecreasing sequence $x_1 \lesssim_{(\Gamma)} x_2 \lesssim_{(\Gamma)} \cdots$ in X such that $x_{n+1} \in S(x_n)$ and

$$-\,d(x_n, x_{n+1}) \leq \tau(x_n) + \frac{1}{2^n}. \tag{3.2}$$

By (iii), we have $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$. Then there exists $n_2 \in \mathbb{N}$, such that $d(x_n, x_{n+1}) < \frac{\varepsilon}{8}$ whenever $n \in \mathbb{N}$ with $n \geq n_2$. On the other hand, since $S(x_{n+1}) \subseteq S(x_n)$ for all $n \in \mathbb{N}$, we have,

$$\begin{aligned}\tau(x_n) &= \inf_{y \in S(x_n)} [-d(x_n, y)] \\ &\leq \inf_{y \in S(x_{n+1})} [-d(x_n, y)] \\ &\leq \inf_{y \in S(x_{n+1})} [-d(x_{n+1}, y) + d(x_{n+1}, x_n)] \\ &= \tau(x_{n+1}) + d(x_n, x_{n+1}).\end{aligned}$$

By (3.2), we obtain

$$\tau(x_{n+1}) \geq -2d(x_n, x_{n+1}) - \frac{1}{2^n}, \quad n \in \mathbb{N}.$$

Let $n_3 = \max\{n_1, n_2\}$. Hence,

$$0 \leq -\tau(x_{n+1}) < \frac{\varepsilon}{2} \quad \text{for all} \quad n \geq n_3.$$

Let $y_x = x_{n_3+1}$. Thus $y_x \in S(x)$ and $0 \leq -\tau(y_x) < \frac{\varepsilon}{2}$. If $S(y_x)$ is a singleton, then $\mu_d(S(y_x)) = 0 \leq \varepsilon$. Otherwise, if $u, v \in S(y_x)$, by (3.1), we have

$$\begin{aligned} d(u, v) &\leq d(u, y_x) + d(y_x, v) \\ &\leq -2\tau(y_x) \\ &< \varepsilon, \end{aligned}$$

which implies $\mu_d(S(y_x)) = diam(S(y_x)) \leq \varepsilon$ and S is of type (μ_d). Let $z_1 \lesssim_{(\Gamma)} z_2 \lesssim_{(\Gamma)} \cdots$ be a nondecreasing sequence in X satisfying $\lim_{n\to\infty} \mu_d(\{z_n, z_{n+1}, \ldots\}) = 0$. Hence, $\{z_n\}$ is a nondecreasing Cauchy sequence in X. By the completeness of X, there exists $w \in X$ such that $z_n \to w$ as $n \to \infty$. We want to prove that w is an upper bound of $\{z_n\}_{n=1}^{\infty}$. For each $n \in \mathbb{N}$, since $z_m \in S(z_n)$ for all $m \geq n$ and $z_n \to w$ as $n \to \infty$, by the closedness of $S(z_n)$, we have $w \in S(z_n)$. Hence $z_n \lesssim_{(\Gamma)} w$ for all $n \in \mathbb{N}$ and w is an upper bound of $\{z_n\}$. Therefore $(X, \lesssim_{(\Gamma)}, \mu_d)$ is a μ_d-bounded quasi-ordered set. □

Remark 3.1 The famous Dancš-Hegedüs–Medvegyev's principle [6, 14] can be proved by Theorem 3.3 and the conclusion (ii) of Theorem 3.2.

Using Theorem 3.1, we can also present a simple proof of generalized Altman's principle [1] improved by Kang and Park [10] as follows. Please compare the following proof with Kang and Park's proof in [10]. Recall that an quasi-ordered set $(X, \lesssim)$ is said to be *countably inductive* (in short, a CIO set) if every nondecreasing sequence has an upper bound (cf. [7, 8, 10]).

Theorem 3.4 [10,Theorem 4] *Let $(X, \lesssim)$ be a CIO set and $\ell : X \times X \to (-\infty, \infty]$ a function. Suppose that*

(i) *there exists a function $c : X \to \mathbb{R}$ such that $c(x) \leq \ell(x, y) \leq 0$ for all $x \in X$ and $y \in S(x) := \{y \in X : x \lesssim y\}$;*
(ii) *for any $x \in X$ and $\varepsilon > 0$, there exists $y = y(x, \varepsilon) \in S(x)$ such that $-\varepsilon \leq c(z)$ for all $z \in S(y)$.*

Then for each $x \in X$, there exists $v \in S(x)$ such that $\ell(v, z) = 0$ for all $z \in S(v)$ (i.e., $v \in S(x)$ is a ℓ-maximal element; see [10]).

Proof Define a sizing-up function $\mu_\ell : 2^X \to [0, \infty]$ by

$$\mu_\ell(A) = \begin{cases} 0, & \text{if } A = \emptyset; \\ \sup\{-\ell(x, y) : x \in A, y \in S(x)\}, & \text{otherwise}. \end{cases}$$

Then for any $x \in X$ and $\varepsilon > 0$, there exists $y = y(x, \varepsilon) \in S(x)$ such that

$$\begin{aligned} \mu_\ell(S(y)) &= \sup\{-\ell(a, b) : a \in S(y), b \in S(a)\} \\ &\leq \sup\{-c(a) : a \in S(y)\} \\ &\leq \varepsilon. \end{aligned}$$

Hence, S is of type (μ_ℓ). Since every CIO set is a μ_ℓ-bounded quasi-ordered set with respect to any quasi-order $\lesssim$ defined on X, by Theorem 3.1, for each $x \in X$; there exists $v \in X$ such that $x \lesssim v$ and $\mu_\ell(S(v)) = 0$. By the definition of μ_ℓ, for each $x \in X$, there exists $v \in S(x)$ such that $\ell(v, z) = 0$ for all $z \in S(v)$. □

4 Optimization problems and stationary point theorems in uniform spaces and metric spaces

We now present an existence theorem for a μ-bounded quasi-ordered set.

Theorem 4.1 *Let $(X, \lesssim)$ be a quasi-ordered set, $\mathcal{U}$ a Hausdorff uniformity on X, $\kappa : X \to (-\infty, \infty]$ a $\lesssim$-nonincreasing function and $u \in X$ with $\kappa(u) < \infty$. Suppose $\mathcal{M} = \{x \in X : u \lesssim x\}$ is a sequentially $(\mathcal{U}, \lesssim)$-complete subset of X. Define $S_{\mathcal{M}} : \mathcal{M} \to 2^{\mathcal{M}}$ by $S_{\mathcal{M}}(x) = \{y \in \mathcal{M} : x \lesssim y\}$. Assume that $S_{\mathcal{M}}(x)$ is closed for any $x \in \mathcal{M}$ and the following conditions hold.*

(H1) κ is bounded below on $\mathcal{M}$;
(H2) for each $V \in \mathcal{U}$, there exists $\delta = \delta(V) > 0$, such that $x, y \in \mathcal{M}$ with $x \lesssim y$ and $\kappa(x) < \kappa(y) + \delta$ implies $(x, y) \in V$.

Then there exists a sizing-up function $\mu : 2^{\mathcal{M}} \to [0, \infty]$ such that

(a) $S_{\mathcal{M}}$ is of type (μ);
(b) $(\mathcal{M}, \lesssim, \mu)$ is a μ-bounded quasi-ordered set.

Proof Let $\mathcal{D} = \{x \in X : \kappa(x) < \infty\}$. Since $u \in \mathcal{D}$, $\mathcal{D} \neq \emptyset$. For each $x \in \mathcal{M}$, since κ is $\lesssim$-nonincreasing, $\kappa(x) \leq \kappa(u) < \infty$. Hence $\mathcal{M} \subset \mathcal{D}$. For each $x \in \mathcal{M}$, define $\mu_{(x,\kappa)} : 2^{\mathcal{M}} \to [0, \infty]$ by

$$\mu_{(x,\kappa)}(A) = \begin{cases} 0, & \text{if } \ A = \emptyset; \\ \sup\{\left|\kappa(w) - \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)\right| : w \in A\}, & \text{otherwise.} \end{cases}$$

Clearly, $\mu_{(x,\kappa)}$ is a sizing-up function, for all $x \in \mathcal{M}$. We first claim that $S_{\mathcal{M}}$ is of type $(\mu_{(x,\kappa)})$ for all $x \in \mathcal{M}$. Let $x \in \mathcal{M}$ and $\varepsilon > 0$ be given. Then there exists $y_x = y(x, \varepsilon) \in S_{\mathcal{M}}(x)$ such that

$$\kappa(y_x) < \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z) + \varepsilon.$$

For any $a \in S_{\mathcal{M}}(y_x)$, since κ is $\lesssim$ -nonincreasing, we have,

$$0 \leq \kappa(a) - \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z) \leq \kappa(y_x) - \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z) < \varepsilon,$$

which implies $\mu_{(x,\kappa)}(S_{\mathcal{M}}(y_x)) \leq \varepsilon$ and hence $S_{\mathcal{M}}$ is of type $(\mu_{(x,\kappa)})$. Next, we will verify that for each $x \in \mathcal{M}$, $(\mathcal{M}, \lesssim, \mu_{(x,\kappa)})$ is a $\mu_{(x,\kappa)}$-bounded quasi-ordered set. Let $x \in \mathcal{M}$ be given and let $c_1 \lesssim c_2 \lesssim \cdots \lesssim c_n \lesssim c_{n+1} \lesssim \cdots$ be a nondecreasing sequence in $\mathcal{M}$ satisfying

$$\begin{aligned} 0 &= \lim_{n\to\infty} \mu_{(x,\kappa)}(\{c_n, c_{n+1}, \ldots\}) \\ &= \lim_{n\to\infty} \sup\left\{\left|\kappa(w) - \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)\right| : w \in \{c_n, c_{n+1}, \cdots\}\right\}. \end{aligned}$$

So it follows that $\lim_{n\to\infty} \kappa(c_n) = \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)$. Let $V \in \mathcal{U}$ and choose $W \in \mathcal{U}$ such that $W \circ W^{-1} \subset V$. Thus, by (H2), there exists $\delta = \delta(W) > 0$, such that $x, y \in \mathcal{M}$ with $x \lesssim y$, and $\kappa(x) < \kappa(y) + \delta$ implies $(x, y) \in W$. Let $\zeta_x := \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)$. Then there exists $n_0 \in \mathbb{N}$ such that $\zeta_x - \frac{1}{2}\delta \leq \kappa(c_n) < \zeta_x + \frac{1}{2}\delta$ for all $n \in \mathbb{N}$ with $n \geq n_0$. For $m, n \in \mathbb{N}$ with $m \geq n \geq n_0$, we have,

$$\kappa(c_n) - \kappa(c_m) \leq \kappa(c_n) - \zeta_x + \frac{1}{2}\delta < \delta,$$

which implies $(c_n, c_m) \in W$ and hence $(c_m, c_n) \in W^{-1}$. Since $W \circ W^{-1} \subset V$, we have $(c_n, c_m) \in V$ and $(c_m, c_n) \in V$ for $m \geq n \geq n_0$. Therefore, $\{c_n\}_{n \in \mathbb{N}}$ is a nondecreasing $(\mathcal{U})$-Cauchy sequence in $\mathcal{M}$. By the sequentially $(\mathcal{U}, \lesssim)$-completeness of $\mathcal{M}$, there exists $w \in \mathcal{M}$, such that $c_n \to w$ as $n \to \infty$. For each $n \in \mathbb{N}$, since $S_{\mathcal{M}}(c_n)$ is closed and

$$c_m \in S_{\mathcal{M}}(c_m) \subseteq S_{\mathcal{M}}(c_n) \quad \text{for all } m \geq n,$$

we obtain $w \in S_{\mathcal{M}}(c_n)$ or $c_n \lesssim w$. So w is an upper bound of $\{c_n\}$. Therefore for each $x \in \mathcal{M}$, $(\mathcal{M}, \lesssim, \mu_{(x,\kappa)})$ is a $\mu_{(x,\kappa)}$-bounded quasi-ordered set. The proof is completed. □

Theorem 4.2 *Under the same hypothesis as in Theorem 4.1, for each $x \in \mathcal{M}$, there exists $v_x \in \mathcal{M}$ such that*

(*a*) $\kappa(v_x) = \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)$;
(*b*) $S_{\mathcal{M}}(v_x) = \{v_x\}$;
(*c*) v_x *is a maximal element of* $\mathcal{M}$.

Proof Applying Theorems 3.1 and 4.1, for each $x \in \mathcal{M}$, there exists $v_x \in \mathcal{M}$ such that $x \lesssim v_x$ and $\mu_{(x,\kappa)}(S_{\mathcal{M}}(v_x)) = 0$. Since $v_x \in S(v_x)$, $\left|\kappa(v_x) - \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)\right| \leq \mu_{(x,\kappa)}(S_M(v_x)) = 0$. Therefore $\kappa(v_x) = \inf_{z \in S_{\mathcal{M}}(x)} \kappa(z)$ and (a) is proved. To see (b), since $S_{\mathcal{M}}(v_x) \subseteq S_{\mathcal{M}}(x)$ and $\kappa(v_x) \leq \kappa(z)$ for all $z \in S_{\mathcal{M}}(x)$, we have $\kappa(v_x) < \kappa(z) + \delta$ for all $z \in S_{\mathcal{M}}(v_x)$ and all $\delta > 0$. Hence, by (H2), $(v_x, z) \in V$ for all $z \in S_{\mathcal{M}}(v_x)$ and all $V \in \mathcal{U}$. Since $\mathcal{U}$ is a Hausdorff uniformity, $S_{\mathcal{M}}(v_x) = \{v_x\}$. Finally, we prove conclusion (c). If $v_x \lesssim \xi$ for some $\xi \in \mathcal{M}$, then $\xi \in S_{\mathcal{M}}(v_x) = \{v_x\}$. Hence $v_x = \xi$ and v_x is a maximal element of $\mathcal{M}$. □

Theorem 4.3 *Let $(X, \mathcal{U})$ be a Hausdorff uniform space, $\varphi : X \to (-\infty, \infty]$ a l.s.c. function and $u \in X$ with $\varphi(u) < \infty$. Let $\mathcal{M} = \{x \in X : \varphi(x) \leq \varphi(u)\}$ be a sequentially $(\mathcal{U})$-complete subset of X, and $T : X \to 2^{\mathcal{M}}$ a multivalued map with nonempty values. Suppose that*

(i) φ *is bounded below on* $\mathcal{M}$;
(ii) *for each $x \in X$, there exists $y_x \in T(x)$ such that $\varphi(y_x) \leq \varphi(x)$;*
(iii) *for each $V \in \mathcal{U}$, there exists $\delta = \delta(V) > 0$, such that $x, y \in \mathcal{M}$ with $\varphi(y) \leq \varphi(x) < \varphi(y) + \delta$ implies $(x, y) \in V$.*

Then there exists $v \in \mathcal{M}$ such that

(1) $\varphi(v) \leq \varphi(u)$;
(2) $v \in \mathcal{F}(T)$;
(3) $\varphi(v) = \inf_{z \in \mathcal{M}} \varphi(z)$;

Proof Define a quasi-order $\lesssim_{(\varphi)}$ on X by

$$x \lesssim_{(\varphi)} y \iff \varphi(y) \leq \varphi(x).$$

Then $\varphi : X \to (-\infty, \infty]$ is a $\lesssim_{(\varphi)}$ -nonincreasing function. Let $S_{\mathcal{M}} : \mathcal{M} \to 2^{\mathcal{M}}$ be defined by

$$\begin{aligned} S_{\mathcal{M}}(x) &= \{y \in \mathcal{M} : x \lesssim_{(\varphi)} y\} \\ &= \{y \in \mathcal{M} : \varphi(y) \leq \varphi(x)\}. \end{aligned}$$

It follows from the lower semicontinuity of φ that $S_{\mathcal{M}}(x)$ is closed for any $x \in \mathcal{M}$. It is easy to see that all the conditions of Theorem 4.2 are also satisfied. Therefore, by Theorem 4.2, there exists $v = v(u) \in \mathcal{M} = \mathcal{S}_{\mathcal{M}(u)}$ such that the conclusions (1) and (3) hold. Let us prove (2). By (ii), there exists $z_v \in T(v)$ such that $\varphi(z_v) \le \varphi(v)$. So we have

$$z_v \in S_{\mathcal{M}}(v) = \{v\},$$

which implies $v = z_v \in T(v)$ or equivalently, $v \in \mathcal{F}(T)$. The proof is completed. □

The following conclusion is immediate from Theorem 4.3.

Theorem 4.4 *Let $(X, \mathcal{U})$, φ, u, $\mathcal{M}$ and conditions (i) and (iii) be the same as in Theorem 4.3. If we assume that $T : X \to \mathcal{M}$ is a single-valued map satisfying $\varphi(Tx) \le \varphi(x)$ for all $x \in X$, then the conclusions of Theorem 4.3 hold.*

Remark 4.1 In [17], Valyi had proved a uniform space version of Dancš–Hegedüs–Medvegyev's principle and gave some applications. Note that Theorems 4.3 and 4.4 are proved in this paper by applying Theorems 3.1 and 4.1 (or 4.2) without the detour of using Dancš–Hegedüs–Medvegyev's principle.

As another consequence of Theorem 4.3, we have the following existence theorem in metric spaces.

Theorem 4.5 *Let (X, d) be a complete metric space $u \in X$, $f : X \to (-\infty, \infty]$ be a l.s.c. function and $f(u) < \infty$. Let $\mathcal{M} = \{x \in X : f(x) \le f(u)\}$ and $T : X \to 2^{\mathcal{M}}$ a multivalued map with nonempty values. Suppose that*

- (i) *f is bounded below on $\mathcal{M}$;*
- (ii) *for each $x \in X$, there exists $y_x \in T(x)$ such that $f(y_x) \le f(x)$;*
- (iii) *for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$, such that $x, y \in \mathcal{M}$ with $f(y) \le f(x) < f(y) + \delta$ implies $d(x, y) < \varepsilon$.*

Then the conclusions of Theorem 4.3 hold.

Proof It follows from the lower semicontinuity of f that $\mathcal{M}$ is closed in X, hence complete. For each $\varepsilon > 0$, let

$$V(\varepsilon) = \{(x, y) \in X \times X : d(x, y) < \varepsilon\}.$$

It is easy to see that the family $\mathcal{U}_d = \{V(\varepsilon) : \varepsilon > 0\}$ is a Hausdorff uniformity on X and $\mathcal{M}$ is $(\mathcal{U}_d)$-complete. Therefore the results follow from Theorem 4.3. □

Finally, we establish a system of nonconvex equilibrium theorem in compact metric spaces.

Theorem 4.6 (System of nonconvex equilibrium theorem) *Let I be a finite index set. For each $i \in I$, let (X_i, d_i) be a compact metric space and let $X = \prod_{i \in I} X_i$ with the metric $d(x, y) = \sup_{i \in I} d_i(x_i, y_i)$, where $x = (x_i)_{i \in I}$, $y = (y_i)_{i \in I} \in X$. For each $i \in I$, suppose that the function $F_i : X \times X_i \to \mathbb{R}$ satisfies the following assumptions:*

- (i) *$F_i(x, x_i) = 0$ for all $x = (x_i)_{i \in I} \in X$;*
- (ii) *$F_i(x, z_i) \le F_i(x, y_i) + F_i(y, z_i)$ for all $x, y, z \in X$;*
- (iii) *for each $x \in X$, $y_i \to F_i(x, y_i)$ is l.s.c. and bounded below;*
- (iv) *for each $y_i \in X_i$, $x \to F_i(x, y_i)$ is u.s.c.*

Then there exists $x_0 \in X$ such that for each $i \in I$, $F_i(x_0, y_i) \ge 0$ for all $y_i \in X_i$.

Proof Applying [7,Theorem 3.4], for each $(n, i) \in \mathbb{N} \times I$, there exists $x^n \in X$ such that $F_i(x^n, y_i) + \frac{1}{n} d_i(x_i^n, y_i) \geq 0$ for all $y_i \in X_i$. By the compactness of X, there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ and $x_0 \in X$ such that $x^{n_k} \to x_0$ as $k \to \infty$. For each $i \in I$, let $y_i \in X_i$ be fixed. By (iv), we have

$$\begin{aligned} F_i(x_0, y_i) &\geq \limsup_{k\to\infty} F_i(x^{n_k}, y_i) + \limsup_{k\to\infty} \left(\frac{1}{n_k} d_i \left(x_i^{n_k}, y_i \right) \right) \\ &\geq \limsup_{k\to\infty} \left(F_i(x^{n_k}, y_i) + \frac{1}{n_k} d_i \left(x_i^{n_k}, y_i \right) \right) \geq 0. \end{aligned}$$

Since $y_i \in X_i$ is arbitrary, $F_i(x_0, y_i) \geq 0$ for all $y_i \in X_i$. □

Remark 4.2 Theorem 4.6 generalizes and improves Propositions 3.2 and 3.3 in [2].

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