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Variational relation problems and equivalent forms of generalized Fan-Browder fixed point theorem with applications to Stampacchia equilibrium problems

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Abstract In this paper, we study the existence theorems of solution for variational relation problems. From the existence theorems of solution for variational relation problems, we study equivalent forms of generalized Fan-Browder fixed point theorem, existence theorems of solutions for Stampacchia vector equilibrium problems and generalized Stampacchia vector equilibrium problems. Our results contains many orginal results and have many applications in Nonlinear Analysis.

Keywords Fan-Browder fixed point theorem · Variational relation problem · Generalized variational relation problem · Stampacchia equilibrium problem

1 Introduction

In 1968, Browder [1] established the following celebrated fixed point theorem:

Theorem A (Browder) *Let X be a nonempty compact convex subset of a topological vector space (in short t.v.s.), $F : X \multimap X$ be a multivalued map. Suppose that*

(i) *$F(x)$ is a nonempty convex subset of X for each $x \in X$;*
(ii) *$F^{-}(y)$ is open in X for each $y \in X$.*

Then there exists $\bar{x} \in X$ such that $\bar{x} \in F(\bar{x})$.

Browder fixed point theorem is a powerful tool in Mathematics, it has many applications, generalizations and has many equivalent forms. See Ref. [2] and references therein.

Let X be a set, $R(x, y)$ be a relation linking elements $x \in X$ and $y \in X$. A variational relation problem is formulated as follows:
Find $\bar{x} \in X$ such that $R(\bar{x}, y)$ holds for all $y \in X$.

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Typical instances of variational relation problems are the following:

(i) *variational inclusion problem* (see Lin and Tu [3], Lin and Chuang [4] and references therein);
(ii) *equilibrium problem* (see Blum and Oettli [5]);
(iii) *Ekeland's variational principle* (see Lin and Chuang [4]);
(iv) *KKM theorem and nonempty intersection theorem* (see Yuan [2]);
(v) *maximal element theorem and fixed point theorems* (see Yuan [2]);
(vi) *section properties* (see Yuan [2]).

One can see [6–11] for details.

Luc [6], Luc, Sarabi and Soubeyran [7], first studied the existence of solution for variational relation problem. Lin and Ansari [8], Lin and Wang [9], Balaj and Lin [10] obtained existence theorems of solutions for variational relation problems and gave their applications to fixed point theorems, variational inclusion problems and systems of nonempty intersection theorem.

Let X be a nonempty subset of a topological vector space (in short t.v.s.) E, Z be a topological space, $R(x,z)$ be a relation linking $x \in X$ and $z \in Z$ and $\rho(x,z,w)$ be a relation linking $x \in X$, $z \in Z$ and $w \in Y$. Let $H, S : X \multimap Z$, $F : X \times Z \times Y \multimap V$, $G : X \times Z \multimap V$, $T : X \times Z \multimap Y$, be multivalued maps, Y and V be t.v.s., $C : X \multimap V$ be a multivalued map such the for each $x \in X$, $C(x)$ is a nonempty closed convex cone in V. Throughout this paper, we use these notations unless specified otherwise. In this paper, we study the following generalized maximal element problem:

(GMP) Find $\bar{x} \in X$ such that $H(\bar{x}) \cap S(\bar{x}) = \emptyset$.

From the existence theorem of solution for (GMP), we study the following existence theorems of solution for variational relation problems.

(VR) Find $\bar{x} \in X$ such that $R(\bar{x}, z)$ does not hold for all $z \in S(\bar{x})$.

From the existence theorem of solution for (VR), we study the following Stampacchia type vector equilibrium problem.

(GSVEP1) Find $\bar{x} \in X$ such that $F(\bar{x}, z, w) \not\subseteq (-C(\bar{x})\backslash\{0\})$ for all $z \in S(\bar{x})$ and $w \in T(\bar{x}, z)$.
(GSVEP2) Find $\bar{x} \in X$ such that $F(\bar{x}, z, w) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$ for all $z \in S(\bar{x})$ and $w \in T(\bar{x}, z)$.
(GSVEP3) Find $\bar{x} \in X$ such that for each $z \in S(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $F(\bar{x}, z, w) \not\subseteq -C(\bar{x})\backslash\{0\}$.
(GSVEP4) Find $\bar{x} \in X$ such that for each $z \in S(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $F(\bar{x}, z, w) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$.

The special cases of (GSVEP1) and (GSVEP2) are the following Stampacchia vector equilibrium problems.

(SVEP1) Find $\bar{x} \in X$ such that $G(\bar{x}, z) \not\subseteq -C(\bar{x})\backslash\{0\}$ for all $z \in S(\bar{x})$.

(SVEP2) Find $\bar{x} \in X$ such that $G(\bar{x}, z) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$ for all $z \in S(\bar{x})$.

To the best of our knowledge, there are very few results on generalized Stampacchia type vector equilibrium problems (GSVEP1-4) and Stampacchia vector equilibrium problems (SVEP1) and (SVEP2). Recently Fu [12], Wang et al. [13], Lin et al. [14,15] studied Generalized Stampacchia vector equilibrium problems and Stampacchia vector equilibrium problems. But our approach and results are different from Fu [12], Wang et al. [13], Lin et al. [14,15], Pardalos et al.[16], Giannessi et al.[17], Mauger et al. [18], Lalitha [19].

In this paper, we apply a generalized Fan-Browder fixed point theorem to study the existence theorems of solution for generalized maximal element problem, variational relations problem. We show the equivalent theorems between the existence theorems of solution for variational relation problems, generalized maximal element theorem and Fan-Browder fixed point theorem. We also apply the existence theorems of solution for variational relation problems to study the existence theorems of solution for generalized Stampacchia equilibrium problems, Stampacchia equilibrium problems, Our results are different from any existence results in the literature and will have many applications in nonlinear analysis.

2 Preliminaries

Let X and Y be nonempty sets. A multivalued mapping (or simply, a mapping) $T : X \multimap Y$ is a function from X into the power set of Y. To a mapping $T : X \multimap Y$, we define $T^* : Y \multimap X$ by $T^-(y) = \{x \in X : y \in T(x)\}$, and $T^*(y) = X \backslash T^-(y)$. The set $T^-(y)$ is called the fiber of T on y and $T^*(y)$ is called cofiber of T on y. For $A \subseteq X$, $T(A) = \cup_{x \in A} T(x)$ is the images of A under T. If X and Y are topological spaces, a mapping $T : X \multimap Y$ is said to be: (i) upper semicontinuous (in short, u.s.c.) (respectively, lower semicontinuous (in short, l.s.c.))if for every closed subset B of Y, the set $\{x \in X : T(x) \cap B \neq \emptyset\}$ (respectively, $\{x \in X : T(x) \subseteq B\}$) is closed; (ii) continuous if it is u.s.c. and l.s.c.; (iii) closed if its graph (that is, the set $GrT = \{(x, y) \in X \times Y, y \in T(x), x \in X\}$) is a closed subset of $X \times Y$.

For a set X, $\langle X \rangle$ will denote the collection of finite subsets of X.

Throughout this paper, all topological spaces are assumed to be Hausdorff. The following lemma collects know facts about u.s.c. or l.s.c. mappings (see, for instance, [20] for assertion (i), [21] for assertion (ii) and [22] for assertion (iii)).

Lemma 2.1 *Let X and Y be topological spaces and $T : X \multimap Y$ be a mapping.*

- (i) *If T is closed and Y is compact, then T is u.s.c.*
- (ii) *If T has compact values, then T is a upper semicontinuous if and only if for every net $\{x_t\}$ in X converging to $x \in X$ and for any net $\{y_t\}$ with $y_t \in T(x_t)$, there exist $y \in T(x)$ and a subnet $\{y_{t_\alpha}\}$ of $\{y_t\}$ converging to y.*
- (iii) *T is l.s.c. if and only if for any $x \in X$, $y \in T(x)$ and any net $\{x_t\}_{t \in \Lambda}$ converging to x, there exists a net $\{y_t\}$ converging to y, with $y_t \in T(x_t)$ for each $t \in \Lambda$.*

The following Lemma is a particular case of Proposition 2 *in* [23].

Lemma 2.2 *Let X be a nonempty convex subset of a t.v.s. E, Z be a topological space. The following assertions are equivalent*

- (i) *S has convex cofibers;*
- (ii) *for each nonempty finite subset N of X, $S(coN) \subseteq S(N)$.*

The following Lemma is a special case of Theorem 4 [23].

Theorem 2.1 (Generalized Fan Browder fixed point theorem) *Let X be a closed convex subset of a t.v.s., E, A, $X \multimap X$ be a multivalued map satisfying the following conditions:*

- (i) *$A(x)$ is a nonempty convex subset of X for each $x \in X$;*
- (ii) *$A^-(y)$ is open for each $y \in X$;*
- (iii) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that*

$$D \backslash K \subseteq \cup\{A^-(y) : y \in D\}.$$

Then there exists $\bar{x} \in X$ such that $\bar{x} \in A(\bar{x})$.

Definition 1 Let Y and Z be convex sets in vector spaces. A mapping $Q : Y \multimap Z$ is said to be quasiconcave if $Q(y_1) \subseteq B$ and $Q(y_2) \subseteq B$ implies that $Q(\lambda y_1 + (1-\lambda)y_2) \subseteq B$ for all convex set $B \subseteq Z$, $y_1, y_2 \in Y$ and $0 \leq \lambda \leq 1$.

3 Existence theorems of variational relation problems

Theorem 3.1 *Let X be a nonemtpy closed convex subset of a t.v..s, E, Z be a compact topological space. Suppose that*

(i) *$S(x)$ is a closed set for each $x \in X$ and H is a closed multiavlued map;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $H(x) \cap S(y) = \emptyset$.*
(iii) *for each $x \in X$, there set $\{y \in X : H(x) \cap S(y) = \emptyset\}$ is convex;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ such that for all $z \in S(y)$, $z \notin H(x)$.*

Then there exists $\bar{x} \in X$ such that $H(\bar{x}) \cap S(\bar{x}) = \emptyset$.

Proof Let $A : X \multimap X$ be defined by

$$A(x) = \{y \in X : H(x) \cap S(y) = \emptyset\}.$$

Since H is closed and Z is a compact set, by Lemma 2.1, H is u.s.c. By (i), each $y \in X$, $S(y)$ is a closed set and

$$A^{-}(y) = \{x \in X : H(x) \cap S(y) = \emptyset\}$$

is open in X.
By (ii), for each $x \in X$, $A(x) \neq \emptyset$.
By (iii), for each $x \in X$, $A(x)$ is a convex set.
By (iv), for each $x \in D \backslash K$, there exists $y \in D$ such that $x \in A^{-}(y)$. That is $D \backslash K \subset U_{y \in D} A^{-}(y)$.

Then it follows from Theorem 2.1, there exists $\bar{x} \in X$ such that $\bar{x} \in A(\bar{x})$. Therefore $H(\bar{x}) \cap S(\bar{x}) = \emptyset$. □

Remark 3.1 In Theorem 3.1, if $H(x_1) = \emptyset$ for some $x_1 \in X$, then $H(x_1) \cap S(x_1) = \emptyset$, and Theorem 3.1 holds.

Remark 3.2 Theorem 2.1 and 3.1 are equivalent.

Proof We see Theorem 2.1 implies Theorem 3.1. Suppose that all the conditions of Theorem 2.1 are satisfied. Let $S, H : X \multimap X$ be defined by $H(x) = \{x\}$ and $A^{-}(y) = X \backslash S(y)$. Then H is closed and for each $y \in X$, $S(y)$ is closed. Since $A(x) \neq \emptyset$ for each $x \in X$, there exists $y \in X$ such that $x \in A^{-}(y)$. Then for each $x \in X$, there exists $y \in X$ such that

$$H(x) \cap S(y) = \{x\} \cap S(y) \subseteq A^{-}(y) \cap S(y) = \emptyset.$$

For each $x \in X$,

$$\begin{aligned} \{y \in X : H(x) \cap S(y) = \emptyset\} &= \{y \in X : x \notin S(y)\} \\ &= \{y \in X : x \in A^{-}(y)\} = A(x) \text{ is convex.} \end{aligned}$$

By (iii), for each $x \in D\backslash K$, there exists $y \in X$ such that $H(x) \cap S(y) = \emptyset$.

Then by Theorem 3.1, there exists $\bar{x} \in X$ such that $H(\bar{x}) \cap S(\bar{x}) = \emptyset$. Therefore $\bar{x} \in A(\bar{x})$ and Theorem 2.1 follows.

By Theorem 3.1, we have the following existence theorem of solution for variational relation problem. □

Theorem 3.2 *Let X be a nonempty closed convex subset of a t.v.s. E, Z be a compact topological space. Suppose that*

(i) *R is closed in both variables and $S(x)$ is a closed subset of Z for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $R(x, z)$ does not hold for all $z \in S(y)$;*
(iii) *for each $x \in X$,*
$\{y \in X : R(x, z)$ does not hold for all $z \in S(y)\}$ is convex;
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that for each $z \in S(y)$, $R(x, z)$ does not hold.*

Then there exists $\bar{x} \in X$ such that $R(\bar{x}, z)$ does not hold for all $z \in S(\bar{x})$.

Proof Let $H : X \multimap X$ be defined by

$$H(x) = \{z \in Z : R(x, z) \text{ holds}\}.$$

Then H is closed. Indeed, let $(x, z) \in \overline{Gr H}$, then there exists a net $\{(x_\alpha, z_\alpha)\}_{\alpha \in \Lambda}$ in $Gr H$ such that $R(x_\alpha, z_\alpha)$ holds, $z_\alpha \in Z$ and $(x_\alpha, z_\alpha) \to (x, z)$. Since Z is a closed set and R is closed in the first and second variables, we see $z \in Z$, $R(x, z)$ holds, and $(x, z) \in Gr H$. Therefore $Gr H$ is a closed set and H is closed. By (ii), for each $x \in X$, there exists $y \in X$ such that $H(x) \cap S(y) = \emptyset$. By (iii), for each $x \in X$, $\{y \in X : H(x) \cap S(y) = \emptyset\}$ is convex. By (iv), for each $x \in D\backslash K$, there exists $y \in D$ such that for each $z \in S(y)$, $z \notin H(x)$.

Then by Theorem 3.1, there exists $\bar{x} \in X$ such that $R(\bar{x}, z)$ does not hold for all $z \in S(\bar{x})$. □

Remark 3.3 Theorems 3.1 and 3.2 are equivalent.

Proof We see Theorem 3.1 $\Rightarrow$ Theorem 3.2. Now suppose that all the conditions of Theorem 3.2 are satisfied. Let the relation R be defined by $R(x, z)$ holds $\Leftrightarrow$ $z \in H(x)$.

Condition (i) of Theorem 3.1 implies R is closed in both variables. Condition (ii) of Theorem 3.1, implies condition (ii) of Theorem 3.2. Conditions (iii) and (iv) of Theorem 3.1 imply conditions (iii) and (iv) of Theorem 3.2 respectively. By Theorem 3.2, there exists $\bar{x} \in X$ such that $R(\bar{x}, z)$ does not hold for all $z \in S(\bar{x})$. Therefore $H(\bar{x}) \cap S(\bar{x}) = \emptyset$ and Theorem 3.1 follows. □

Theorem 3.3 *Theorem 3.2, is true if condition* (iii) *of Theorem 3.2, is replaced by* (iii$'$), *where*

(iii$'$) *One of the following conditions holds:*
(iii$'_a$) *S has convex cofibers;*
(iii$'_b$) *S is quasiconcave and for each $x \in X$, the set $\{z \in Z : R(x, z)$ does not holds$\}$ is convex.*

Proof For each $x \in X$, let $H(x)$ and $M(x)$ be defined by

$$H(x) = \{z \in Z : R(x, z) \text{ holds}\}$$

and

$$M(x) = \{y \in X : H(x) \cap S(y) = \emptyset\}.$$

Then for each $x \in X$, $M(x)$ is convex set. Indeed, let $y_1, y_2 \in M(x)$ and $\lambda \in [0, 1]$.

Then $S(y_i) \subseteq H(x)^c$ for $i = 1, 2$, where $H(x)^c = Z \backslash H(x)$. If (iii$'_a$) holds, by Lemma 2.2,

$$S(\lambda y_1 + (1 - \lambda)y_2) \subseteq S(y_1) \cup S(y_2) \subseteq H(x)^c.$$

Then $H(x) \cap S(\lambda y_1 + (1 - \lambda)y_2) = \emptyset$ and $\lambda y_1 + (1 - \lambda)y_2 \in M(x)$. If (iii$'_b$) holds, then $H(x)^c$ is a convex set.
Since S is quasiconcave,
$S(\lambda y_1 + (1 - \lambda)y_2) \subseteq H(x)^c$ and
$H(x) \cap S(\lambda y_1 + (1 - \lambda)y_2) = \emptyset$. Therefore
$\lambda y_1 + (1 - \lambda)y_2 \in M(x)$ and $M(x)$ is a convex set for each $x \in X$ and Theorem 3.3 follows from Theorem 3.1.

As applications of Theorem 3.2, we study the following existence theorems of solution for generalized variational relation problems. □

Theorem 3.4 *Let X be a nonempty closed convex subset of a t.v.s., E, Z be a compact topological space. Let Y be a topological space. Suppose that*

(i) *ρ is closed in all variables, T is l.s.c. and $S(x)$ is a closed subset of Z for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that for each $z \in S(y)$, there exists $w \in T(x, z)$ such that $\rho(x, z, w)$ does not hold;*
(iii) *for each $x \in X$, the set*
$\{y \in X :$ for each $z \in S(y)$, there exists $w \in T(x, z)$ such that $\rho(x, z, w)$ does not hold$\}$ is convex.
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that for each $z \in S(y)$, there exists $w \in T(x, z)$ such that $\rho(x, z, w)$ does not hold.*

Then there exists $\bar{x} \in X$ such that for each $z \in S(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $\rho(\bar{x}, z, w)$ does not hold.

Proof Let the relation R be defined by

$$R(x, z) \text{ holds } \Leftrightarrow \rho(x, z, w) \text{ holds } \quad \text{for all } w \in T(x, z).$$

Then we follow the same argument as in Theorem 3.4 [8], we can show that R is closed in both variables. Then Theorem 3.4 follows from Theorem 3.2. □

Theorem 3.5 *Let X, Z, Y and S be the same as in Theorem* 3.4. Suppose that

(i) ρ is closed in all variables, T is a u.s.c. multivalued map with nonempty compact values and $S(x)$ is a closed subset of Z for each $x \in X$;
(ii) for each $x \in X$, there exists $y \in X$ such that for each $z \in S(x)$, $w \in T(x, z)$, $\rho(x, z, w)$ does not hold;
(iii) for each $x \in X$, the set
$\{y \in X : \rho(x, z, w)$ does not hold for all $z \in S(y)$ and $w \in T(x, z)\}$ is convex;
(iv) there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that $\rho(x, z, w)$ does not hold for all $z \in S(y)$ and $w \in T(x, z)$.

Then there exists $\bar{x} \in X$ such that $\rho(\bar{x}, z, w)$ does not hold for each $z \in S(\bar{x})$ and $w \in T(\bar{x}, z)$.

Proof Let relation R be defined by

$$R(x, z) \text{ hold } \Leftrightarrow \rho(x, z, w) \text{ holds for some } w \in T(x, z).$$

Then R is closed in the first and second variables. Indeed, if $\{(x_\alpha, z_\alpha)\}_{\alpha\in\Lambda}$ be any net in $X \times Z$ such that $(x_\alpha, z_\alpha) \to (x, z)$, then $\rho(x_\alpha, z_\alpha, w_\alpha)$ holds for some $w_\alpha \in T(x_\alpha, z_\alpha)$. By (i) and Lemma 2.1, there exists $z \in T(x, z)$ and $\{w_\alpha\}_{\alpha\in\Lambda}$ has a subnet $\{w_{\alpha_\lambda}\}_{\alpha_\lambda\in A}$ such that $w_{\alpha_\lambda} \to w$. We see $\rho(x_{\alpha_\lambda}, z_{\alpha_\lambda}, w_{\alpha_\lambda})$ holds. By (i), $\rho(x, z, w)$ holds for some $x \in T(x, z)$. Therefore ρ is closed in all the variables. Then Theorem 3.5, follows from Theorem 3.2. □

Theorem 3.6 *Let X be a nonempty closed convex subset of a t.v.s. E, Z be a compact topological space. Suppose that*

(i) *R is closed in the second variable and H is a closed multivalued map;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $R(y, z)$ does not hold for all $z \in H(x)$;*
(iii) *for each $x \in X$ and $z \in H(x)$,*
$\{y \in X : R(y, z)$ does not hold for all $z \in H(x)\}$ is convex;
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ and for each $z \in H(x)$, $R(y, z)$ does not hold.*

Then there exists $\bar{x} \in X$ such that $R(\bar{x}, z)$ does not holds for all $z \in H(\bar{x})$.

Proof Let $S : X \multimap Z$ be defined by

$$S(y) = \{z \in Z : R(y, z) \text{ holds}\}.$$

By (i), $S(y)$ is a closed set for each $y \in X$.
By (iii), for each $x \in X$,

$$\begin{aligned}&\{y \in X : R(y, z) \text{ does not holds for all } z \in H(x)\}\\ &\quad = \{y \in X : S(y) \cap H(x) = \emptyset\}\end{aligned}$$

By (iv), for each $x \in X\backslash K$, there exists $y \in D$ such that

$$H(x) \cap S(y) = \emptyset.$$

Then by Theorem 3.1, there exists $\bar{x} \in X$ such that $H(\bar{x}) \cap S(\bar{x}) = \emptyset$. Therefore $R(\bar{x}, z)$ does not hold for all $z \in H(\bar{x})$. □

Remark 3.4 Theorems 3.1, 3.2 and 3.6 are equivalent.

Proof We see that Theorem 3.1 ⇒ Theorem 3.6. Suppose that all conditions of Theorem 3.1 are satisfied. Let R be defined by

$$R(x, z) \text{ holds } \Leftrightarrow z \in S(x).$$

By (i) of Theorem 3.1, for each $x \in X$, $R(x, \cdot)$ is closed in the second variable. By (ii), for each $x \in X$, there exists $y \in X$ such that $R(y, z)$ does not hold for all $z \in H(x)$. Conditions (iii) and (iv) of Theorem 3.1 imply conditions (iii) and (iv) of Theorem 3.6 respectively. Then by Theorem 3.6, we prove Theorem 3.1. Therefore Theorem 3.1 and 3.6 are equivalent. We have shown that Theorem 3.1 ⇔ Theorem 3.2. □

Remark 3.5 In Theorem 3.2, R is closed in all variables and $S(y)$ is a closed set for each $y \in X$, but in Theorem 3.6, R is closed in the second variable and H is closed, the other conditions of these two theorems are also different.

Theorem 3.7 *Let X, E, Z and H be the same as in Theorem 3.6. Let Y be a topological space. Suppose that*

(i) *ρ is closed in the second and third variables, H is a closed multivalued map and T is a l.s.c. multivalued map;*
(ii) *for each $x \in X$, there exists $y \in X$ such that for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $\rho(y, z, w)$ does not hold;*
(iii) *for each $x \in X$, the set $\{y \in X :$ for each $z \in H(x)$ there exists $w \in T(x, z)$ such that $\rho(y, z, w)$ does not hold$\}$ is convex;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ and for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $\rho(y, z, w)$ does not hold.*

Then there exists $\bar{x} \in X$ such that for each $z \in H(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $\rho(\bar{x}, z, w)$ does not hold.

Proof We apply Theorem 3.7 and follow the same argument as in Theorem 3.4, we can prove Theorem 3.7.

Theorem 3.8 *In Theorem 3.7, if conditions* (i), (ii), (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′). *Then there exists $\bar{x} \in X$ such that $\rho(\bar{x}, z, w)$ does not hold for all $z \in H(\bar{x})$ and $w \in T(\bar{x}, z)$, where*

(i′) *ρ is closed in the second and third variables, H is a closed multivalued map and T is a u.s.c. multivalued map with nonempty compact values;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that $\rho(y, z, w)$ does not hold for all $z \in H(x)$, $w \in T(x, z)$.*
(iii′) *for each $x \in X$, the set $\{y \in X : \rho(y, z, w)$ does not hold for all $z \in H(x)$ and $w \in T(x, z)\}$ is convex.*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ for which $\rho(y, z, w)$ does not hold for all $z \in H(x)$ and $w \in T(x, z)$.*

Then there exists $\bar{x} \in X$ such that for each $\rho(\bar{x}, z, w)$ does not holds for all $z \in H(\bar{x})$ and $w \in T(\bar{x}, z)$.

Proof We apply Theorem 3.6 and follow the same argument as in Theorem 3.5, we can prove Theorem 3.8. □

Theorem 3.9 *Theorems 3.1, 3.2 3.3, 3.4, 3.6, 3.7, 3.8 are equivalent to generalized Fan-Browder fixed point theorem* (*Theorem 2.1*).

Proof We see from Remarks 3.2, 3.3, 3.4, Theorem 2.1 ⇔ Theorem 3.1 ⇔ Theorem 3.6 ⇔ 3.2. We see Theorem 3.2 ⇒ Theorem 3.4. Theorem 3.2 ⇒ 3.5.

It suffices to prove Theorem 3.8 ⇒ Theorem 3.6 and Theorem 3.7 ⇒ Theorem 3.6, and Theorems 3.4, 3.5 ⇒ 3.2.

Sppose all conditions of Theorem 3.6, let $T(z, w) = \{w_0\}$ for some $w_0 \in Y$ and $\rho(t, z, w) = R(x, z)$ for all $(x, z, w) \in X \times Z \times Y$. Then by Theorem 3.7 or Theorem 3.8, we can prove Theorem 3.6. Similarly, we can prove Theorem 3.4 ⇒ 3.2, Theorem 3.5 ⇒ Theorem 3.2. □

Definition 3 [10] Let Z and V be nonempty convex sets in two vector spaces and $M \subseteq Z \times V$. We say that M is (z, v) convex if for each $(z_1, v_1), (z_2, v_2) \in M$ and for any $z \in [z_1, z_2]$, there exists $v \in [v_1, v_2]$ such that $(z, v) \in M$, where $[z_1, z_2] = \{z : \cdot z = \lambda z_1 + (1 - \lambda) z_2,\ 0 \leq \lambda \leq 1\}$.

Remark 3.6 Condition (iii) of Theorem 3.4 is satisfied if one of the following conditions holds:

(iii'_a) S has convex cofibers;
(iii'_b) S is quasiconcave and for each $x \in X$, $z_1, z_2 \in Z$, $0 \leq \lambda \leq 1$, $\lambda T(x, z_1) + (1 - \lambda) T(x, z_2) \subseteq T(x, \lambda z_1 + (1 - \lambda) z_2)$, and the set

$$N(x) = \{(z, v) \in Z \times V : \rho(x, z, v) \text{ does not hold}\} \text{ is } (z, v) - \text{ convex.}$$

Proof For each $x \in X$. Let $H(x)$ and $M(x)$ be defined by $H(x)^c = \{z \in Z$: there exists $v \in T(x, z)$ such that $\rho(x, z, v)$ does not hold$\}$ and $H(x) = Z \backslash H(x)^c$ and $M(x) = \{y \in X : H(x) \cap S(y) = \emptyset\}$.

If (iii'_b) holds, then $H(x)^c$ is a convex set. Indeed, let $z_1, z_2 \in H(x)^c$ and $0 \leq \lambda \leq 1$. Then there exists $w_1 \in T(x, z_1)$ and $v_2 \in T(x, z_2)$ such that $\rho(x, z_1, v_1)$ and $\rho(x, z_2, w_2)$ do not hold. Hence (z_1, v_1) $(z_2, v_2) \in N(x)$. Since $N(x)$ is (z, v)-convex, for any $z \in [z_1, z_2]$, there exists $v \in [v_1, v_2]$ such that $\rho(x, z, v)$ does not hold.

$$v = \lambda v_1 + (1 - \lambda) v_2 \in \lambda T(x, z_1) + (1 - \lambda) T(x, z_1) \subseteq T(x, \lambda z_1 + (1 - \lambda) z_2) = T(x, z).$$

This shows that $H(x)^c$ is a convex set. Then we follow the same argument as in Theorem 3.3, we prove Remark 3.6. □

4 Generalized Stampacchia type vector equilibrium problem

As applications of Theorems 3.4 and 3.5, we study the existence theorems of solution for generalized Stampacchia vector equilibrium problems.

Theorem 4.1 *Let X, Z, Y and S be the same as in Theorem 3.5. Let V be a t.v.s. Suppose that*

(i) *F is l.s.c., T is u.s.c. multivalued map with nonempty compact values, C is closed and $S(x)$ is a closed set in Z for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $F(x, z, w) \not\subseteq -C(x)$ for all $z \in S(y)$ and all $w \in T(x, z)$;*
(iii) *for each $x \in X$, the set $\{y \in X : F(x, z, w) \not\subseteq -C(x)$ for all $z \in S(y)$ and for all $w \in T(x, z)\}$ is convex.*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ such that $F(x, z, w) \not\subseteq -C(x)$ for all $z \in S(y)$ and all $w \in T(x, z)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, z, w) \not\subseteq -C(\bar{x}) \backslash \{0\}$ for all $z \in S(\bar{x})$ and all $w \in T(\bar{x}, z)$.

Proof Let the relation ρ be defined by

$$\rho(x, z, w) \text{ holds} \Leftrightarrow F(x, z, w) \subseteq -C(x).$$

Then ρ is closed in all variables. Indeed, let $\{(x_\alpha, z_\alpha, w_\alpha)\}_{\alpha\in\Lambda}$ be any net in $X \times Z \times Y$ such that $\rho(x_\alpha, z_\alpha, w_\alpha)$ holds and $(x_\alpha, z_\alpha, w_\alpha) \to (x, z, w)$. Then

$$F(x_\alpha, z_\alpha, w_\alpha) \subseteq -C(x_\alpha) \quad \text{for all } \alpha \in \Lambda.$$

Let $v \in F(x, z, w)$. By (i) and Lemma 2.1, there exists a net $\{v_\alpha\}_{\alpha\in\lambda}$ such that $v_\alpha \in F(x_\alpha, z_\alpha, w_\alpha)$ and $v_\alpha \to v$. One has $v_\alpha \in -C(x_\alpha)$. Since C is closed, $v \in -C(x)$. Therefore $F(x, z, v) \subseteq -C(x)$. This shows that ρ is closed in all variables. All the other conditions of Theorem 3.5 are easily checked. Then by Theorem 3.5. There exists $\bar{x} \in X$ such that $F(\bar{x}, z, w) \not\subseteq -C(\bar{x})$ for each $z \in S(\bar{x})$ and $w \in T(\bar{x}, z)$. Therefore,

$$F(\bar{x}, z, w) \not\subseteq (-C(\bar{x})\backslash\{0\}) \quad \text{for all } z \in S(\bar{x}) \text{ and } w \in T(\bar{x}, z)$$ □

Remark 4.1 By Theorem 3.3, we see that condition (iii) of Theorem 4.1 is satisfied if one of the following conditions is satisfied:

(iii$_a$) S has convex cofibers;
(iii$_b$) S is quasicave and for each $x \in X,\ w \in Y$, each $z_1, z_2 \in Z,\ \lambda \in [0, 1]$ either

$$F(x, z_1, w) \subseteq F(x, \lambda z_1 + (1-\lambda)z_2, w)$$

or

$$F(x, z_2, w) \subseteq F(x, \lambda z_2 + (1-\lambda)z_2, w)$$

Theorem 4.2 *In Theorem 4.1, suppose that*

(i) *F and T are u.s.c. multivalued maps with nonempty compact values, C is closed and $S(x)$ is a closed set in Z for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $F(x, z, w) \cap [-C(x)] = \emptyset$ for each $z \in S(y)$ and $w \in T(x, z)$;*
(iii) *for each $x \in X$, the set $\{y \in X : F(x, z, w) \cap [-C(x)] = \emptyset \text{ for all } z \in S(y) \text{ and for all } w \in T(x, z)\}$ is convex.*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that $F(x, z, w) \cap (-C(x)) = \emptyset$ for all $z \in S(y)$ and all $w \in T(x, z)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, z, w) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$ for all $z \in S(\bar{x})$ and all $w \in T(\bar{x}, z)$.

Proof Let the relation ρ be defined by

$$\rho(x, z, w) \text{ holds } \Leftrightarrow F(x, z, w) \cap (-C(x)) \neq \emptyset.$$

Then ρ is closed in all variables. Indeed, let $\{(x_\alpha, z_\alpha, w_\alpha)\}_{\alpha\in\Lambda}$ be any net in $X \times Z \times W$ such that $(x_\alpha, z_\alpha, w_\alpha) \to (x, z, w)$ and $\rho(x_\alpha, z_\alpha, w_\alpha)$ holds. Then

$$F(x_\alpha, z_\alpha, w_\alpha) \cap (-C(x_\alpha)) \neq \emptyset.$$

Let $v_\alpha \in F(x_\alpha, z_\alpha, w_\alpha) \cap (-C(x_\alpha))$, then $v_\alpha \in F(x_\alpha, z_\alpha, w_\alpha)$. Since F is a u.s.c. multivalued map with nonempty compact values, it follows from Lemma 2.1, $\{v_\alpha\}_{\alpha\in\Lambda}$ has a subnet $\{v_{\alpha_\lambda}\}_{\alpha_\lambda\in\Lambda}$ such that $v_{\alpha_\lambda} \to v \in F(x, z, w)$.

Since $v_{\alpha_\lambda} \in -C(x_{\alpha_\lambda})$ and C is closed, we have $v \in -C(x)$. Therefore $v \in F(x, z, w) \cap (-C(x)) \neq \emptyset$ and $\rho(x, z, w)$ holds. This shows that ρ is closed in all the variables. All other conditions of Theorem 3.5 are easily checked. Then by Theorem 3.5, there exists $\bar{x} \in X$ such that

$$F(\bar{x}, z, w) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset \quad \text{for all } z \in S(\bar{x}) \quad \text{and} \quad w \in T(\bar{x}, z).$$

□

Remark 4.2 By Theorem 3.3, we see that condition (iii) of Theorem 4.2 holds if one of the following conditions hold:

(iii_a) S has convex cofibers;
(iii_b) S is quasiconcave and for each $x \in X,\ w \in Y$, each $z_1, z_2 \in Z,\ \lambda \in [0, 1]$ either

$$F(x, \lambda z_1 + (1-\lambda)z_2, w) \subseteq F(x, z_1, w)$$

or

$$F(x, \lambda z_1 + (1-\lambda)z_2, w) \subseteq F(x, z_2, w)$$

For the special cases of Theorem 4.1 and 4.2, we have the following existence theorems of solution for the Stampacchia type vector equilibrium problems.

Theorem 4.3 *Let X, Z, S be the same as in Theorem 3.5. Let V be a t.v.s.. Suppose that*

(i) *G is l.s.c., C is closed and $S(x)$ is a closed set for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $G(x, z) \not\subseteq -C(x)$ for all $z \in S(y)$;*
(iii) *for each $x \in X$, the set $\{y \in X : G(x, z) \not\subseteq -C(x)$ for all $z \in S(y)\}$ is convex;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that $G(x, z) \not\subseteq -C(x)$ for all $z \in S(y)$.*

Then there exists $\bar{x} \in X$ such that $G(\bar{x}, z) \not\subseteq -C(\bar{x})\backslash\{0\}$ for all $z \in S(\bar{x})$.

Proof Let $y_0 \in Y$, $F : X \times Z \times Y \multimap V$ be defined by $F(x, z, w) = G(x, z)$ and $T : X \times Z \multimap Y$ be defined by $T(x, z) = \{y_0\}$ for all $(x, z) \in X \times Z$.

Then Theorem 4.3 follows from Theorem 4.1. □

Theorem 4.4 *In Theorem 4.3, if conditions* (i), (ii), (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *G is an u.s.c. multivalued map with nonempty compact values, C is closed and $S(x)$ is a closed set for each $x \in X$;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that $G(x, z) \cap (-C(x)) = \emptyset$ for all $z \in S(y)$;*
(iii′) *for each $x \in X$, the set $\{y \in X : G(x, z) \cap (-C(x)) = \emptyset$ for all $z \in S(y)\}$ is a convex set;*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that $G(x, z) \cap (-C(x)) = \emptyset$ for all $z \in S(x)$.*

Then there exists $\bar{x} \in X$ such that $G(\bar{x}, z) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$ for all $z \in S(\bar{x})$.

Proof We apply Theorem 4.2 and follow the same arguments as in Theorem 4.3, we can prove Theorem 4.4. □

Theorem 4.5 *Let X, Z, Y and V be the same as in Theorem 3.4. Suppose that*

(i) *F and T are l.s.c., C is closed and $S(x)$ is a closed set in Z for each $x \in X$;*
(ii) *for each $x \in X$, there exists $y \in X$ such that for each $z \in S(y)$, there exists $w \in T(x,z)$ such that $F(x,z,w) \nsubseteq -C(x)$;*
(iii) *for each $x \in X$, the set $\{y \in X :$ for each $z \in S(y)$, there exists $w \in T(x,z)$ such that $F(x,z,w) \nsubseteq -C(x)\}$ is convex;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that for each $z \in S(y)$, there exists $w \in T(x,z)$ with $F(x,z,w) \nsubseteq -C(x)$.*

Then there exists $\bar{x} \in X$ such that for each $z \in S(\bar{x})$, there exists $w \in T(\bar{x},z)$ with $F(\bar{x},z,w) \nsubseteq -C(\bar{x})\backslash\{0\}$.

Proof We apply Theorem 3.4 and follow similar arguments as in Theorem 4.1, we can prove Theorem 4.5. □

Remark 4.3 Condition (iii) of Theorem 4.5 is satisfied if (iii'_a) or (iii'_b) with "$\rho(x,z,w)$ holds $\Leftrightarrow F(x,z,w) \subseteq -C(x)$" in Remark 3.6 is satisfied.

Theorem 4.6 *In Theorem 4.5, if condition* (i), (ii), (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *F is a u.s.c. multivalued map with nonempty compact values, T is l.s.c., C is a closed map and $S(x)$ is a closed set in Z for each $x \in X$;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that for each $z \in S(y)$, there exists $w \in T(x,z)$ with $F(x,z,w) \cap (-C(x)) = \emptyset$;*
(iii′) *for each $x \in X$, the set $\{y \in X :$ for each $z \in S(y)$, there existrs $w \in T(x,z)$ with $F(x,z,w) \cap (-C(x)) = \emptyset\}$ is convex;*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D\backslash K$, there exists $y \in D$ such that for each $z \in S(y)$, there exists $w \in T(x,z)$ with $F(x,z,w) \cap (-C(x)) = \emptyset$.*

Then there exists $\bar{x} \in X$ such that for each $z \in S(\bar{x})$, there exists $w \in T(\bar{x},z)$ with $F(\bar{x},z,w) \cap (-C(\bar{x})) = \emptyset$.

Proof We apply Theorem 3.5 and follow similar arguments as in Theorem 4.2, we can prove Theorem 4.6. □

Remark 4.4 Condition (iii) in Theorem 4.6 is satisfied if either ($\text{iii}\prime_a$) or ($\text{iii}\prime_b$) with "$\rho(x,z,w)$ holds $\Leftrightarrow F(x,z,w) \cap (-C(x)) \neq \emptyset$" in Remark 3.6 is satisfied.

Theorem 4.7 *Let X be a nonempty closed convex subset of a t.v.s. E, V be t.v.s., Z be a compact topological space. Suppose that*

(i) *for each $x \in X$, $(z,w) \multimap F(x,z,w)$ is l.s.c., $z \multimap T(x,z)$ is a u.s.c. multivalued map with nonempty compact values and H and C are closed multivalued maps;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $F(y,z,w) \nsubseteq -C(x)$ for all $z \in H(x)$, and for all $w \in T(x,z)$*
(iii) *for each $x \in X$, $z \in H(x)$, $w \in T(x,z)$ $\{y \in X : F(y,z,w) \nsubseteq -C(x)\}$ is convex;*

(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \setminus K$, there exists $y \in D$ for which $F(y, z, w) \nsubseteq -C(x)$ for all $z \in H(x)$ and $w \in T(x, z)$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, z, w) \nsubseteq -C(\bar{x}) \setminus \{0\}$ for all $z \in H(\bar{x})$ and $w \in T(\bar{x}, z)$.

Proof We apply Theorem 3.8 and follow the same arguments as in Theorem 4.1, we can prove Theorem 4.7. □

Remark 4.5 Condition (iii) of Theorem 4.7 is satisfied if for each $y_1, y_2 \in X$, $\lambda \in [0, 1]$, either $F(y_1, z, w) \subseteq F(\lambda y_1 + (1 - \lambda) y_2, z, w)$ or $F(y_2, z, w) \subseteq F(\lambda y_1 + (1 - \lambda) y_2, z, w)$.

Theorem 4.8 *In Theorem 4.7, if conditions* (i), (ii), (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *for each $x \in X$, $(z, w) \multimap F(x, z, w)$ is l.s.c., $z \multimap T(x, z)$ is l.s.c., H and C are closed;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that for each $z \in H(x)$ there exists $w \in T(x, z)$ such that $F(y, z, w) \nsubseteq -C(x)$;*
(iii′) *for each $x \in X$, the set $\{y \in X :$ for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $F(y, z, w) \nsubseteq -C(x)\}$ is convex;*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \setminus K$, there exists $y \in D$ and for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $F(y, z, w) \nsubseteq -C(x)$.*

Then there exists $\bar{x} \in X$ such that for each $z \in H(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $F(\bar{x}, z, w) \nsubseteq -C(\bar{x}) \setminus \{0\}$.

Proof We apply Theorem 3.7 and follow the same arguments as in Theorem 4.5, we can prove Theorem 4.8. □

Remark 4.6 Condition (iii) of Theorem 4.8 is satisfied if condition either (iii$'_a$) or (iii$'_b$) (iii$' + b$) with "$\rho(x, z, w)$ holds $\Leftrightarrow$ $F(x, z, w) \subseteq -C(x)$" in Remark 3.6 is satisfied.

Theorem 4.9 *In Theorem 4.7, if conditions* (i), (ii), (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *for each $x \in X$, $(z, w) \multimap F(x, z, w)$ is a u.s.c. multivalued map with nonempty compact values, T is a u.s.c. multivalued map with compact values, H and C are closed;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that $F(y, z, w) \cap (-C(x)) = \emptyset$ for all $z \in H(x)$ and $w \in T(x, z)$;*
(iii′) *for each $x \in X$, each $z \in H(x)$, $w \in T(x, z)$, the set $\{y \in X : F(y, z, w) \cap (-C(x)) = \emptyset\}$ is convex;*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \setminus K$, there exists $y \in D$ and for each $z \in H(x)$, and each $w \in T(x, z)$, $F(y, z, w) \cap (-C(x)) = \emptyset$.*

Then there exists $\bar{x} \in X$ such that $F(\bar{x}, z, w) \cap (-C(\bar{x}) \setminus \{0\}) = \emptyset$ for all $z \in H(\bar{x})$ and $w \in T(\bar{x}, z)$.

Proof We apply Theorem 3.6 and follow the same arguments as in Theorem 4.2, we can prove Theorem 4.9. □

Theorem 4.10 *In Theorem 4.7, if conditions* (i)*,* (ii)*,* (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *for each $x \in X$, $(z, w) \multimap F(x, z, w)$ is a u.s.c. multivalued map with nonempty compact values and T is a l.s.c. multivalued map, H and C are closed;*
(ii′) *for each $x \in X$, there exists $y \in X$ such that for each $z \in H(x)$, there exists $w \in T(x, z)$ for which*

$$F(y, z, w) \cap (-C(x)) = \emptyset;$$

(iii′) *for each $x \in X$, the set* *$\{y \in X$: for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $F(y, z, w) \cap (-C(x)) = \emptyset\}$ is convex.*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ and for each $z \in H(x)$, there exists $w \in T(x, z)$ such that $F(y, z, w) \cap (-C(x)) = \emptyset$.*

Then there exists $\bar{x} \in X$ such that for each $z \in H(\bar{x})$, there exists $w \in T(\bar{x}, z)$ such that $F(\bar{x}, z, w) \cap (-C(x) \backslash \{0\}) = \emptyset$.

Proof We apply Theorem 3.7 and follow the same arguments as in Theorem 4.6, we can prove Theorem 4.10. □

Remark 4.7 (a) Condition (iii) of Theorem 4.9 is satisfied if for each $y_1, y_2 \in X$, $\lambda \in [0, 1]$, $z \in H(x)$, $w \in T(x, z)$.

$$F(\lambda y_1 + (1-\lambda)y_2, z, w) \subseteq F(y_1, z, w)$$

or

$$F(\lambda y_1 + (1-\lambda)y_2, z, w) \subseteq F(y_2, z, w).$$

For the special cases of Theorem 4.7 and 4.8, we have the following Theorem.

Remark 4.8 Condition (iii) of Theorem 4.10 is satisfied if condition (iii$'_b$) with $\rho(x, z, w)$ holds $\Leftrightarrow$ $F(x, z, w) \cap (-C(x)) = \emptyset$ is Remark 3.6 is satisfied.

Theorem 4.11 *Let X, Z, V, C and H be the same as in Theorem 4.7. Suppose that*

(i) *for each $x \in X$, $z \multimap G(x, z)$ is l.s.c., H and C are closed;*
(ii) *for each $x \in X$, there exists $y \in X$ such that $G(y, z) \nsubseteq -C(x)$ for all $z \in H(x)$;*
(iii) *for each $x \in X$, $z \in H(x)$,* *$\{y \in X : G(y, z) \nsubseteq -C(x)\}$ is convex;*
(iv) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each $x \in D \backslash K$, there exists $y \in D$ for which* *$G(y, z) \nsubseteq -C(x)$ for all $z \in H(x)$.*

Then there exists $\bar{x} \in X$ such that $G(\bar{x}, z) \nsubseteq -C(\bar{x}) \backslash \{0\}$.

For the special case of Theorem 4.9 and 4.10, we have the following theorem.

Theorem 4.12 *In Theorem 4.11, if conditions* (i)*,* (ii)*,* (iii) *and* (iv) *are replaced by* (i′), (ii′), (iii′) *and* (iv′) *respectively, where*

(i′) *for each* $x \in X$, $z \multimap G(x, z)$ *is u.s.c., H and C are closed;*
(ii′) *for each* $x \in X$, *there exists* $y \in X$ *such that for each* $z \in H(x)$*;* $G(y, z) \cap (-C(x)) = \emptyset$.
(iii′) *for each* $x \in X$, $z \in H(x)$, $\{y \in X : G(y, z) \cap (-C(x)) = \emptyset\}$ *is convex;*
(iv′) *there exist a nonempty compact subset K of X and a nonempty compact convex subset D of X such that for each* $x \in D\backslash K$, *there exists* $y \in D$ *for which* $G(y, z) \cap (-C(x)) = \emptyset$ *for all* $z \in H(x)$.

Then there exists $\bar{x} \in X$ *such that* $G(\bar{x}, z) \cap (-C(\bar{x})\backslash\{0\}) = \emptyset$ *for all* $z \in H(\bar{x})$.

References

1. Browder, F.E.: The fixed point theory of multi-valued mappings in topological vector spaces. Math. Ann. **177**, 283–301 (1968)
2. Yuan, G.X.Z.: KKM theory and applications in Nonlinear Anlayais. Marcel Dekker Inc., New York (1999)
3. Lin, L.J., Tu, C.I.: The studies of variational inclusions problems and variational disclusions problems with applications. Nonlinear Anal. TMA **69**, 1981–1998 (2008)
4. Lin, L.J., Chuang, C.S.: Systems of nonempty intersection theorems with applications. Nonlinear Anal. TMA **69**, 4063–4073 (2008)
5. Blum, E., Oettli, W.: From optimization and variational inequalities to equilibrium problems. Math. Student **63**, 123–146 (1994)
6. Luc, D.T.: An abstract problem in variational analysis. J. Optim. Theory Appl. **138**, 65–78 (2008)
7. Luc, D.T., Sarabi, E., Soubeyran, A.: Existence of solutions in variational relations without convexity. J. Math. Anal. Appl. **384**, 544–555 (2010)
8. Lin, L.J., Anasri, Q.H.: Systems of variational relation problems with applications. Nonlinear Anal. **72**, 1210–1220 (2010)
9. Lin, L.J., Wang, S.Y.: Simultaneous variational relations problems and related applications. Comput. Math. Appl. **58**, 1711–1721 (2009)
10. Balaj, M., Lin, L.J.: Equivalent forms of generalized KKM theorem and their applications. Nonlinear Anal. **23**, 673–682 (2010)
11. Balaj, M., Lin, L.J.: Variational relation problems with applications. J. Optim. Theory Appl. **148**, 1–13 (2011)
12. Fu, J.Y.: Stampacchia generalized vector quasi-equilibrium problem and vector saddle points. J. Optim. Theory Appl. **128**, 259–268 (2006)
13. Wang, S.H., Fu, J.Y.: Stampacchia generalized vector quasi-equilibrium problems with set-valued mapping. J. Glob. Optim. **44**, 99–110 (2009)
14. Lin, L.J., Yu, Z.T., Kassay, G.: Eixstence of equilibrium for multivalued mappings and its applications to vector equilibria. J. Optim. Theory Appl. **134**, 189–208 (2002)
15. Lin, L.J., Chuang, C.S., Wang, S.Y.: From quasivariational inclusion problems to Stampacchia vector quasiequilibrium problems., Stampacchia set-valued vector Ekeland's variational principle and Caristi's fixed point theorem, nonlinear analysis, theory. Methods Appl. **71**, 179–185 (2009)
16. Pardalos, P.M., Rassias, T.M., Khan, A.A.: Nonlinear Analysis and Variational Problems in Honor of George Isac. Series: Springer Optimization and its Applications, Vol. 35, Springer (2010)
17. Giannessi, F., Pardalos, M A., Panos, M.: Equilibrium Problems:Nonsmooth Optimization and Variational Inequality Models. Series: Nonconvex Optimization and Its Applications, Vol. 58, Springer (2002)
18. Maugeri, A., Raciti, F.: On general infinite diminsenal complementarity problems. Optim. Lett. **2** (2008)
19. Lalitha, C.S.: A note on duality of generalized equilibrium problem. Optim. Lett. **4**, 57–66 (2010)
20. Aubin, J.P., Cellina, A.: Differential inclusions. Springer, Heidelberg (1941)
21. Ferro, F.: A minimax theorem for vector-valued functions. J. Optim. Theory Appl. **60**, 19–31 (1989)
22. Tan, N.X.: Quasi-variational inequalities in linear locally convex Hausdorff spaces. Math. Nach. **122**, 231–245 (1995)
23. Lin, L.J.: Applications of a fixed point theorem in *G*-convex space. Nonlinear Anal. **46**, 601–608 (2001)