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A NUMERICAL INVESTIGATION OF THE CRITICAL PHASE OF RANDOM BOOLEAN NETWORKS

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Our numerical simulation first displays exponential increase of the mean number of attractors with N for $K = 2, 3, 4$ and $50 \leqq N \leqq 350$. The mean length of attractors also increases exponentially with N for $K = 2$, but increases linearly with N for $K = 3, 4$. We further found the larger the K the larger S/N value; which yields the results that the mean length and the mean number of attractors of critical random Boolean networks will decrease with larger K.

Keywords: Random Boolean networks; Kauffman model; network dynamics.

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1. Introduction

Random Boolean networks were proposed in 1969 by S. A. Kauffman who has since developed an important model of how life began.[1] The Kauffman model describes a system of randomly connected nodes with dynamics based on Boolean update functions. It has been very successful in its applications in biophysics, statistical physics, etc.

In the dynamic system of the model the $N^{1/2}$ behavior of the mean number and length of attractors has piqued the interest of by biologists to investigate deeper. Due to increasing computer power, it was recently discovered that the old assumption about $N^{1/2}$ behavior of the mean number of attractors for critical $K = 2$ Kauffman networks was wrong. Also with increasing computer power, several discrepancies of the mean number of attractors were seen: an increase faster than $N^{1/2}$ in Ref. 2; stretched exponential in Refs. 3 and 4, linear in Ref. 5; faster than linear in Ref. 6. Then, in a beautiful analytical study, Samuelsson and Troein,[7] and Kauffman *et al.*[8] proved that the number of attractors grows indeed faster than any power law with network size N. On the other hand, Bastolla and Parisi,[2] and Bhattacharjya and Liang[9] found that probability distributions of the length of attractors are very broad (power-law like) and become broader with system size.

This means that the mean length of attractors increases much faster than $N^{1/2}$. However, Kauffman *et al.*,[8] Drossel *et al.*,[10] and Drossel[11] claim that the number of attractors of length L is Poisson distributed with a mean $1/L$ and that the mean length of attractors increases with network size N faster than any power law.

Since the mean number and length of attractors of Kauffman networks are affected mainly by stable core (abbreviated S) or relevant elements (abbreviated R_E) investigation of the properties of stable core and relevant elements is very important. The behavior of the stable nodes was first studied by Flybjerg.[12] Almost all the number of stable nodes of core can be found.[5] These nodes do not take part in the asymptotic dynamics. Relevant elements are defined as those elements whose states are not constant and control at least one relevant element.[3] These elements completely determine the number and length of attractors. Socolar and Kauffman[6] found numerically that for critical $K = 2$ networks the mean number of nonfrozen nodes $\langle N\text{–}S\rangle$ scales as $N^{2/3}$ and the mean number of relevant elements scales as $N^{1/3}$. The same result is hidden in the analytical work on attractor numbers by Samuelsson and Troein,[7] as shown in Ref. 10. The aim of our investigation is to further clarify the dependence of these properties on the network of size N.

2. Method of Investigation

The random Boolean network consists of N Boolean elements, $\{\sigma_1, \sigma_2, \ldots, \sigma_N\}$, with σ_j being either zero or one. The value of each element σ_j at time $t+1$ is determined by the values at time t of K other elements $\sigma_{j1}, \sigma_{j2}, \ldots, \sigma_{jK}$, which are called the controlling elements for element σ_j. Once the connections among the elements in a system are established, each element σ_j is assigned with a Boolean update function f_j of its K controlling elements. In other words, the network dynamics for the system is given by the equation

$$\sigma_j(t+1) = f_j(\sigma_{j1}(t), \sigma_{j2}(t), \ldots, \sigma_{jK}(t)), \qquad \text{for } j = 1, 2, \ldots, N\,. \tag{1}$$

Our main purpose of this investigation is to explore the properties of Kauffman's deterministic dynamics networks at critical line. These properties include mean number of attractors $\langle N_C\rangle$, mean length of attractors $\langle L_C\rangle$, mean number of stable cores $\langle N_{SC}\rangle$, mean number of relevant elements $\langle R_E\rangle$, etc. Accordingly, our numerical method of implementation consists of the following:

1. Using the formula

$$K_c = \frac{1}{2p(1-p)}$$

   to determine the phase of a network, where K_c is the critical K value[12] and p is the probability for the state 0 of Boolean update functions $f_j(\sigma_1, \sigma_2, \ldots, \sigma_K)$.
2. Connections K and Boolean update functions are randomly taken at the beginning and kept fixed afterwards. On each discrete time step, the outputs are

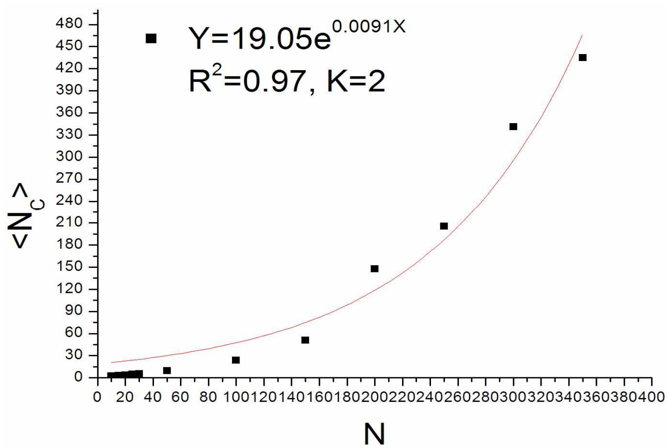
updated synchronously. Because connections K and Boolean update functions are randomly taken at the beginning and kept fixed afterwards the network system is deterministic and any initial state will finally result in a cycle.

3. The initial state of the system configuration is randomly taken until a cycle is found.
4. The network reduction algorithm form is implemented[5]; each network is allowed to evolve for one hour (approximately $2^{23} \sim 2^{26}$ initial states chosen) and then stopped, and L_C, N_C, the number of stable cores (S), and the number of relevant elements (R_E) of each network recorded.
5. For each $50 \leqq N \leqq 350$, 2000 network calculations for same K and p were implemented.

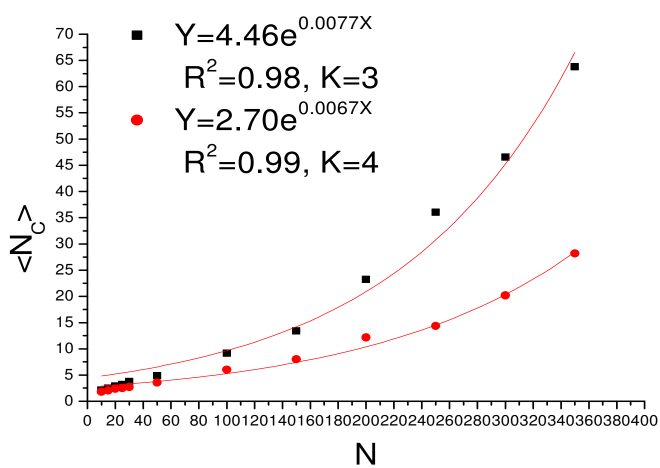
3. Simulated Results and Discussion

For the critical random Boolean network at $K = 2, 3, 4$, via our long-time numerical simulation we have obtained various properties of the networks for $50 \leqq N \leqq 350$. The results are as follows:

1. Figures 1(a) and 1(b) show respectively the behavior of the mean number of attractors $\langle N_C \rangle$ versus N. When $N \geqq 50$ the increases of $\langle N_C \rangle$ with N for the three different K's appear to be consistent with exponential, which conforms with the derived results of Samuelsson[7] and Kauffman.[8]
2. Figures 2 and 3 show respectively the different behavior of the mean length of attractors $\langle L_C \rangle$ versus N. For $K = 2$, Fig. 2 shows exponential increase of $\langle L_C \rangle$ with N, which is consistent with the expected results of Kauffman.[8] Further, one can easily see from our simulated results that the distribution of L_C, which is somewhat like a power law, tends to agree the viewpoints of Bastolla and Parisi *et al.*[2,9] As can be seen in Fig. 4 the averaged frequency (probability distribution) of L_C is quite broad, in which the large portion of probability is located in a small region near $L_C = 50$. Contrary to Fig. 2, Fig. 3 for $K = 3, 4$ shows respectively linear increase of $\langle L_C \rangle$ with N. The difference between the exponential increase as shown in Fig. 2 and the linear increase as shown in Fig. 3 is, as we suggest, due to the increasing behavior of the mean number of stable cores $\langle S \rangle$ with increasing N as shown in Fig. 5 below.
3. Figure 5 shows the behavior of the mean number of nonfrozen elements $\langle N\text{–}S \rangle$ versus N for $50 \leqq N \leqq 350$ and $K = 2, 3, 4$ respectively. For $10 \leqq N \leqq 350$ the $\langle N\text{–}S \rangle$ is seen to be approximately proportional to N whereas Bastolla claims the $N^{3/4}$ proportion when $N \to \infty$. Yet Socolar and Samuelsson *et al.* maintain that $\langle N\text{–}S \rangle$ is proportional to $N^{2/3}$ when $N \to \infty$. The possible reason for the discrepancies would be that the N values used in our numerical simulation were not large enough.
4. Figure 6 shows the behavior of the mean number of stable cores $\langle S \rangle$ versus N for $K = 2, 3, 4$. $\langle S \rangle$ is seen to increase linearly with N. These results are in



(a)



(b)

Fig. 1. The mean number of attractors $\langle N_C \rangle$ versus N for the number of controlling elements $K = 2, 3, 4$ in the critical phase.

consistency with those obtained by Bilke *et al.*,[5] and are also consistent with the results derived by Flyvbjerg.[12]

5. Figure 7 shows the log–log plot of the mean number of relevant elements $\langle R_E \rangle$ versus N for $K = 2, 3, 4$. $\langle R_E \rangle$ is seen to increase proportional to $N^{4/5}$ for $10 \leqq N \leqq 350$. Yet Bastolla claims the $N^{1/2}$ dependence of $\langle R_E \rangle$ as $N \to \infty$.[3] However, Socolar[6] and Kauffmen *et al.*[8] hold that the relation between $\langle R_E \rangle$ and N is not definite; $\langle R_E \rangle$ is proportional to $N^{1/3}$ when $N \to \infty$, as can be seen in Fig. 2 of Ref. 8.

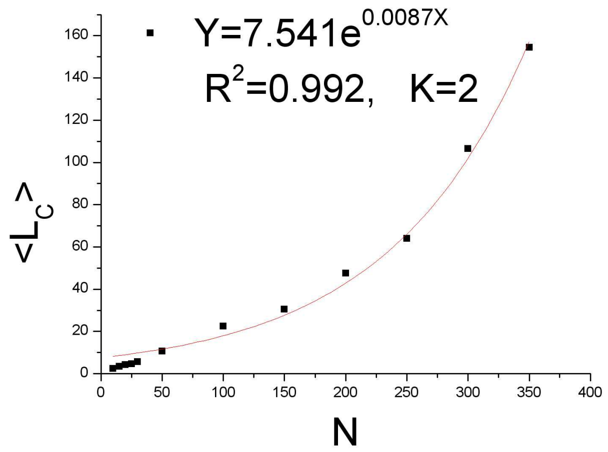


Fig. 2. The exponential behavior of the mean length of attractors $\langle L_C \rangle$ versus N for the number of controlling elements $K = 2$ in the critical phase.

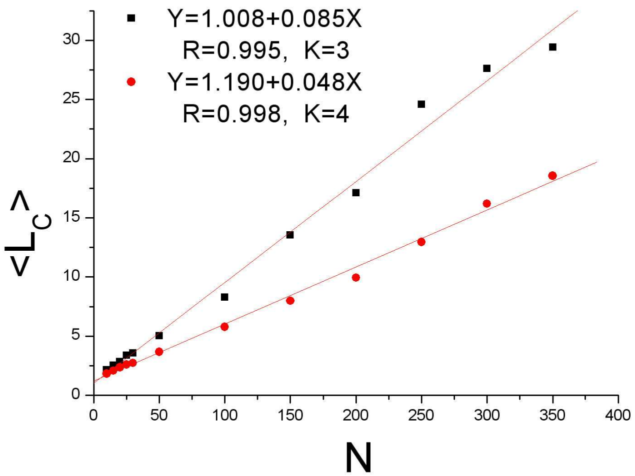


Fig. 3. The linear behavior of the mean length of attractors $\langle L_C \rangle$ versus N for the number of controlling elements $K = 3, 4$ in the critical phase.

4. Conclusion

The Kauffman model for random Boolean networks is essentially a model for biologically genetic regulatory systems. Nevertheless the model has attracted the interest of physicists in the 1980s, due to its analogy with the disordered systems studied in statistical mechanics, such as the mean field spin glass. Consequently various properties in the critical phase of the networks are very important. Thus numerical investigation of various properties in the critical phase is worth exploration. The long-time numerical simulation for $50 \leqq N \leqq 350$ now obtains the following results

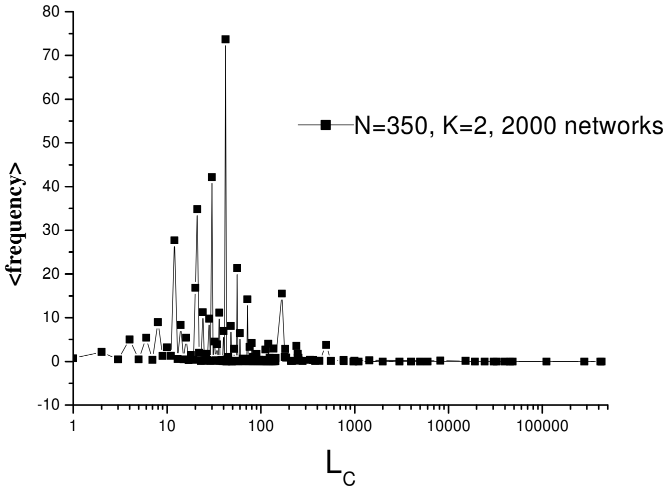


Fig. 4. The average frequency of the length of attractors L_C for 2000 networks $N = 350$, $K = 2$.

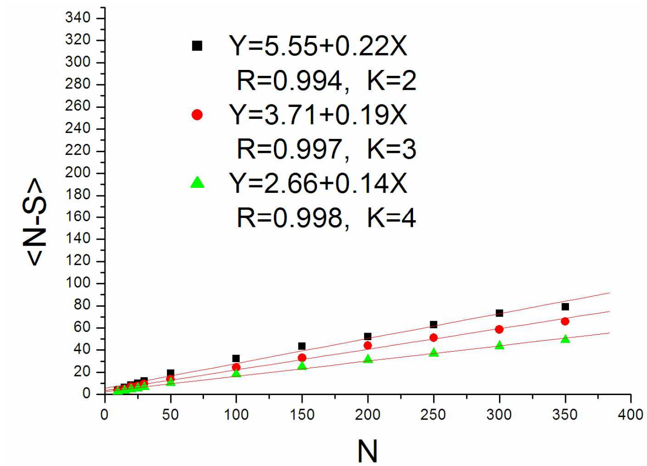


Fig. 5. The log–log plot of the mean nonfrozen elements $\langle N\text{–}S\rangle$ versus N for the number of controlling elements $K = 2, 3, 4$ in the critical phase.

in conclusion:

1. For $K = 1, 2, 3, 4$, when $50 \leqq N \leqq 350$ the mean number of attractors $\langle N_C\rangle$ increases exponentially with N, which already conforms with the theoretical behavior of $\langle N_C\rangle$ when $N \to \infty$. (For $K = 1$ $\langle N_C\rangle$ has been analytically shown to increase exponentially with N.[11])
2. We have first discovered via numerical simulation that for $K = 1, 2$, and when $50 \leqq N \leqq 350$, the mean length of attractors $\langle L_C\rangle$ already increases exponentially with N, which is already in conformity with the theoretical behavior of

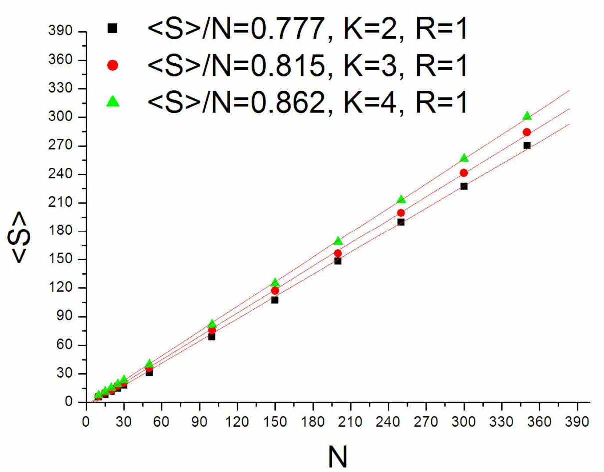


Fig. 6. The mean number of stable cores $\langle S \rangle$ versus N for the number of controlling elements $K = 2, 3, 4$ in the critical phase.

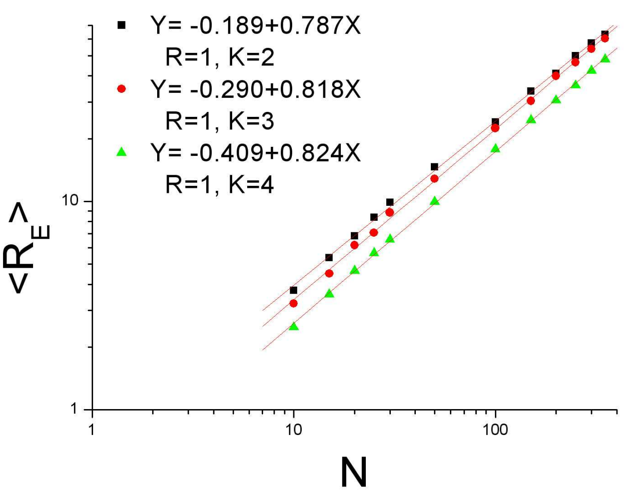


Fig. 7. The log–log plot of the mean number of relevant elements $\langle R_E \rangle$ versus N for the number of controlling elements $K = 2, 3, 4$ in the critical phase.

$\langle N_C \rangle$ when $N \to \infty$. But for $K = 3, 4$, $\langle L_C \rangle$ increases linearly with N. (For $K = 1$, $\langle L_C \rangle$ has been analytically shown to have the same exponential dependence with N.)[11]

3. For $K = 2, 3, 4$, the mean number of stable cores $\langle S \rangle$ increases linearly with N and we first obtained the result: the larger the K the larger $\langle S \rangle / N$. When N is fixed, the larger K goes with the larger $\langle S \rangle$ which leads to smaller $\langle L_C \rangle$ and $\langle N_C \rangle$, as is clearly seen in Figs. 1–3.

4. For $50 \leqq N \leqq 350$, the mean number of relevant elements $\langle R_E \rangle$ is approximately proportional to $N^{4/5}$.

In conclusion, based on our long computing time numerical simulation and the theoretical derivation of Kauffman *et al.*, it should be confirmed that for the critical $K = 2$ random Boolean networks both the mean number and the length of attractors would increase exponentially with system size N.

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References

1. S. A. Kauffman, *J. Theor. Biol.* **22** (1969) 437.
2. U. Bastolla and G. Parisi, *J. Theor. Biol.* **187** (1997) 117.
3. U. Bastolla and G. Parisi, *Physica D* **115** (1998) 203.
4. U. Bastolla and G. Parisi, *Physica D* **115** (1998) 219.
5. B. Bilke and F. Sjunnesson, *Phys. Rev. E* **65** (2002) 016129.
6. J. E. S. Socolar and S. A. Kauffman, *Phys. Rev. Lett.* **90** (2003) 068702.
7. B. Samuelsson and C. Troein, *Phys. Rev. Lett.* **90** (2003) 098701.
8. V. Kauffman, T. Mihaljev and B. Drossel, *Phys. Rev. E* **72** (2005) 046124.
9. A. Bhattacharjya and S. Liang, *Phys. Rev. Lett.* **77**(8) (1996) 1644.
10. B. Drossel, *Phys. Rev. E* **72** (2005) 016110.
11. B. Drossel, T. Mihaljev and F. Greil, *Phys. Rev. Lett.* **94** (2005) 088701.
12. H. Flyvbjerg, *J. Phys. A* **21** (1988) L955.